Quantum Foundations

Neil Dewar

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Preface

These are the handouts I used for teaching the course “Quantum Foundations” at the MCMP, in the winter semester 2019/20. They aren’t full lecture notes; rather, they are the compressed core of such notes, designed to save the students from having to scribble too much down during class, and to be supplemented by “live” explanation and discussion. But that means that they are, of course, very partial; and especially in the more philosophical parts, they give very simplified descriptions of what’s going on.

Roughly speaking, the notes can be divided into three parts. The first part (Chapters 0–4) are essentially an introduction to the formalism of the theory, from complex numbers through to *-algebras. This formalism is limited to finite-dimensional Hilbert spaces (and *-algebras), in order that mathematical rigour could be combined with simplicity of presentation. (It’s also the reason I talk about *-algebras rather than C*-algebras.) The second part, Chapters 5–8, describes some of the peculiar features of quantum mechanics that differentiate it from classical probability theory. However, all of these features arise from purely “kinematic” considerations, and not from the quantum dynamics. Chapters 9–13 (with the exception of Chapter 11) discuss the philosophical issues posed by quantum dynamics, i.e., the problems that arise from seeking a fully quantum-mechanics treatment of measurement. (In the course of doing which, we end up introducing wavefunctions after all, in the discussion of Bohmian mechanics.) Appendix A describes a game that I came up with to help students understand the proof of the Kochen-Specker theorem. Appendix B is an abridgement of the famous EPR paper, in which all their talk of wavefunctions has been replaced by talk of “state vectors”; some questions for class discussion and analysis are included. Appendix C is some notes on how to calculate correlations for EPR-like states.

At some point, I’d like to revise these notes to both make them more comprehensive, and—more importantly—to check them for errors and notational inconsistencies. But it’s not clear when there will be the time available to do that, so the most I can do is offer a warning to the reader: please do double-check any computations whose results are asserted in this text. And I’d be very grateful to anyone who lets me know of errors they find within.

Munich, February 2020

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1 Chapter 11 (on symmetries in quantum mechanics) is a bit out of place: it is where it is because I had to rearrange a class due to illness.
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0 A primer on complex numbers

0.1 The complex numbers

• The complex numbers $\mathbb{C}$ are an algebraic extension of the real numbers $\mathbb{R}$, used to let us take square roots of negative numbers.

• We define them by first introducing the “imaginary” number $i$, stipulated to be such that $i^2 = -1$, and letting complex numbers be linear combinations (“complexes”) of $i$ and $1$.

• That is, a complex number is anything of the form $a + bi$, where $a, b \in \mathbb{R}$.

• We define addition and multiplication of complex numbers in terms of addition and multiplication of real numbers:

$$(a + bi) + (c + di) := (a + c) + (b + d)i$$  \hspace{1cm} (0.1)

$$(a + bi)(c + di) := (ac - bd) + (bc + ad)i$$  \hspace{1cm} (0.2)

• Note that with these definitions, addition and multiplication are commutative and associative, and multiplication distributes over addition.

• We also introduce the operation of complex conjugation: the complex-conjugate of $z \in \mathbb{C}$ is denoted $\overline{z}$, and conjugation is defined by

$$a + bi := a - bi$$  \hspace{1cm} (0.3)

• One can show that the following identities hold for any $x, y \in \mathbb{C}$:

$$_{\overline{x}} = x$$  \hspace{1cm} (0.4)

$$_{\overline{x + y}} = _{\overline{x}} + _{\overline{y}}$$  \hspace{1cm} (0.5)

$$_{\overline{xy}} = (_{\overline{x}})(_{\overline{y}})$$  \hspace{1cm} (0.6)

• One can also show that $\overline{x} = x$ iff $x \in \mathbb{R}$, and $\overline{x} = -x$ iff $x = bi$ for some $b \in \mathbb{R}$.

• We define the modulus of a complex number by

$$|z| := \sqrt{zz}$$  \hspace{1cm} (0.7)

• Note that if $z = a + bi$ then $\overline{z}z = a^2 + b^2$, so $\overline{z}z$ (and hence $|z|$) is always a positive real number.
• For any complex number $z = a + bi$, if we define the phase $\phi := \tan^{-1}(\frac{b}{a})$, then
  \[ z = |z| \cos \phi + (|z| \sin \phi)i \]  
  (0.8)

• This is referred to as the polar representation of $z$

• Defining $r := |z|$ and $e^{i\phi} := \cos \phi + i \sin \phi$, we often write this as $z = re^{i\phi}$

• Note that multiplying complex numbers is easier in the polar representation:
  \[ (re^{i\phi})(se^{i\psi}) = (rs)e^{i(\phi+\psi)} \]  
  (0.9)

0.2 The complex plane

• It is often convenient to represent the complex numbers on a plane, with the real numbers as the horizontal axis and the imaginary numbers as the vertical axis:

  ![Complex Plane Diagram]

• Complex conjugation corresponds to reflection in the real axis

• The modulus of a complex number $z = re^{i\phi}$ corresponds to its distance from the origin, and the phase to the angle it makes with the real axis:
1 Classical probabilities

1.1 Discrete probability densities

- Let $W$ be a finite set; the elements of $W$ may be thought of as complete, classical configurations of some system or world.

- A (discrete) probability density is a map $\rho : W \to [0, 1]$ such that

$$\sum_{w \in W} \rho(w) = 1 \quad \text{(1.1)}$$

- An event $E$ is defined to be a subset of $W$

- Any event $E$ also has an associated probability, denoted $p(E)$ and defined by

$$p(E) := \sum_{w \in E} p(w) \quad \text{(1.2)}$$

- The map $p$ from subsets of $W$ to probabilities is called a probability measure.
1.2 Random variables

• Given a probability density over a set $W$, a random variable is a map $X : W \to \mathbb{R}$.

• The spectrum $\sigma(X) \subseteq \mathbb{R}$ is the image of $W$ under $X$, i.e.

$$\sigma(X) := \{X(w) \in \mathbb{R} : w \in W\}$$  \hspace{1cm} (1.3)

• There is a natural probability density on the spectrum of $X$, denoted by $\rho_X$ and defined (for $\lambda \in \sigma(X)$) by

$$\rho_X(\lambda) := \sum_{w \in X^{-1}(\lambda)} \rho(w)$$  \hspace{1cm} (1.4)

• More generally, for any $n$ random variables $X_1, \dots, X_n$, there is a discrete probability density on $\sigma(X_1) \times \cdots \times \sigma(X_n)$ given by

$$\rho_{X_1,\ldots,X_n}(\lambda_1,\ldots,\lambda_n) = \sum_{w \in X^{-1}(\lambda)} \rho(w)$$  \hspace{1cm} (1.5)

where $X^{-1}(\lambda) := \bigcap_{i=1}^n X_i^{-1}(\lambda_i)$.

1.3 Expectation values

• Given a random variable $X : W \to \mathbb{R}$, we define the expectation value of $X$ as

$$\mathcal{E}(X) := \sum_{w \in W} X(w)\rho(w)$$  \hspace{1cm} (1.6)

• Let $1_E$ is the characteristic function of $E \subseteq W$, i.e. the function

$$1_E(w) = \begin{cases} 1 & \text{if } w \in E \\ 0 & \text{if } w \notin E \end{cases}$$  \hspace{1cm} (1.7)

• It follows that

$$\mathcal{E}(1_E) = p(E)$$  \hspace{1cm} (1.8)

• Note that $\mathcal{E}$ is linear:

$$\mathcal{E}(X + Y) = \mathcal{E}(X) + \mathcal{E}(Y)$$  \hspace{1cm} (1.9)

$$\mathcal{E}(cX) = c\mathcal{E}(X)$$  \hspace{1cm} (1.10)
2 Hilbert spaces I

2.1 Vector spaces

- The complex conjugate of $\alpha \in \mathbb{C}$ will be denoted $\overline{\alpha}$
- A complex vector space is a set $V$, equipped with
  - a special zero element $0$,
  - an inversion operation $- : V \rightarrow V$,
  - an addition operation $+ : V \times V \rightarrow V$, and
  - a scalar multiplication operation $: \mathbb{C} \times V \rightarrow V$,
such that for any $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$,

\[
\begin{align*}
  u + v &= v + u \quad & (2.1) \\
  u + (v + w) &= (u + v) + w \quad & (2.2) \\
  v + 0 &= v \quad & (2.3) \\
  v + (-v) &= 0 \quad & (2.4) \\
  0 \cdot v &= 0 \quad & (2.5) \\
  1 \cdot v &= v \quad & (2.6) \\
  \alpha \cdot (\beta \cdot v) &= (\alpha \beta) \cdot v \quad & (2.7) \\
  \alpha \cdot (u + v) &= \alpha \cdot u + \alpha \cdot v \quad & (2.8) \\
  (\alpha + \beta) \cdot v &= \alpha \cdot v + \beta \cdot v \quad & (2.9)
\end{align*}
\]

- A real vector space is the same, except that scalar multiplication is over the real numbers, i.e. scalar multiplication is an operation $\cdot : \mathbb{R} \times V \rightarrow V$

- We will abbreviate scalar multiplication $\alpha \cdot v$ by $\alpha v$

- A basis for $V$ is a collection of vectors $v_i \in V$ such that any $v \in V$ can be expressed in the form

\[
v = \sum_i \alpha_i v_i \quad (2.10)
\]

- The dimension of a vector space is the smallest number $n$ such that there exists a basis consisting of $n$ vectors

- A subspace of a vector space $V$ is a nonempty subset $U \subseteq V$ which is closed under addition and scalar multiplication
2.2 Inner products

- Given a complex vector space \( V \), an **inner product** is a binary operation \( \langle \ , \ \rangle : V \times V \rightarrow \mathbb{C} \) such that for any \( u, v, w \in V \) and \( \alpha, \beta \in \mathbb{C} \),
  
  \[
  \langle u, v \rangle = \overline{\langle v, u \rangle} \tag{2.11}
  \]
  
  \[
  \langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle \tag{2.12}
  \]
  
  \[
  \langle u, u \rangle > 0 \text{ if } u \neq 0 \tag{2.13}
  \]

- Given an inner product, the **norm** of any vector \( v \) is defined to be
  
  \[
  |v| := \sqrt{\langle v, v \rangle} \tag{2.14}
  \]

- Two vectors \( u, v \in V \) are **orthogonal** if \( \langle u, v \rangle = 0 \)

- An **orthonormal** basis for \( V \) is a basis consisting entirely of vectors of norm 1, which are all orthogonal to one another

2.3 Hilbert space

- A **finite-dimensional Hilbert space** is a finite-dimensional complex vector space equipped with an inner product

- We use elements of a Hilbert space \( H \) to represent the so-called “pure” states of a quantum system.

- Unlike in the classical case, one cannot determine whether or not the system is in the state represented by \( \psi \) through a single measurement.

- The best one can do is find a measurement setup such that for any state \( \phi \), the probability of a positive result is given by

  \[
  \frac{|\langle \phi, \psi \rangle|^2}{|\phi|^2 |\psi|^2} \tag{2.15}
  \]

- Hence, for any \( \alpha \in \mathbb{C} \) and \( \psi \in H \), the states represented by \( \psi \) and \( \alpha \psi \) cannot be discriminated from one another; we therefore take them to be the same state

- We use this freedom to stipulate that only the **unit norm** elements of a Hilbert space (those such that \( |\psi| = 1 \)) will be used to represent states

- Under this stipulation, the probability of a positive result in the “\( \psi \)-test” is then simply given by \( |\langle \phi, \psi \rangle|^2 \)

- Note that even with this stipulation, there are still multiple elements of \( H \) representing the same state: e.g., \( \psi \) and \( i\psi \).
3 Hilbert spaces II

3.1 Operators

• Given Hilbert spaces $H_1$ and $H_2$, a map $F : H_1 \to H_2$ is linear if

$$F(\alpha\psi + \beta\phi) = \alpha F(\psi) + \beta F(\phi)$$  \hspace{1cm} (3.1)

• Given linear maps $F : H_1 \to H_2$ and $G : H_2 \to H_3$, their composition $GF : \psi \mapsto G(F(\psi))$ is a linear map from $H_1$ to $H_3$

• An operator on a Hilbert space $H$ is a linear map $O : H \to H$

• $\psi \in H$ is an eigenvector of $O$, with eigenvalue $k \in \mathbb{C}$, if

$$O\psi = k\psi$$  \hspace{1cm} (3.2)

• If $\psi$ is an eigenvector, then for any $\alpha \in \mathbb{C}$, $\alpha\psi$ is also an eigenvector with the same eigenvalue; and if $\phi$ and $\psi$ are eigenvectors of the same eigenvalue, so is $\phi + \psi$

• The set of eigenvectors with a given eigenvalue is therefore a subspace, called an eigenspace

• If an eigenvalue has a multi-dimensional eigenspace, it is said to be degenerate

• The trace $\text{Tr}(O)$ of an operator $O$ is the sum of its eigenvalues

• An operator $O$ is symmetric if

$$\langle \phi, O\psi \rangle = \langle O\phi, \psi \rangle$$  \hspace{1cm} (3.3)

• If $O$ is symmetric (and $H$ is of finite dimension), then there exists an orthonormal basis for $H$ consisting of eigenvectors of $O$, all of which have real eigenvalues

• Thus, a symmetric operator provides a decomposition of $H$ into mutually orthogonal eigenspaces, all associated with real eigenvalues

• For any subspace $J \subseteq H$, the projector for $J$ is the (symmetric) operator $P_J$ defined by the condition that for any $\phi \in H$,

$$P_J(\phi) = \begin{cases} 
\phi & \text{if } \phi \in J \\
0 & \text{if } \phi \text{ is orthogonal to } J 
\end{cases}$$  \hspace{1cm} (3.4)
3.2 Quantities and states

- Symmetric operators represent physical quantities: the eigenvalues represent the possible values of the quantity (its spectrum), and a unit eigenvector with eigenvalue \( q \) represents a state which is guaranteed to yield the value \( q \) upon measurement (an eigenstate).

- For example, the projector \( P_J \) represents the quantity which takes the value 1 for all \( \psi \in J \), and the value 0 for all \( \psi \) orthogonal to \( J \).

- If \( \epsilon_q \) is an eigenstate of a non-degenerate eigenvalue \( q \), then for any pure state \( \psi \), the probability of attaining the value \( q \) upon a measurement is given by \( |\langle \psi, \epsilon_q \rangle|^2 \).

- The expectation value of \( Q \) on the state \( \psi \) is given by
  \[
  \langle \psi, Q \psi \rangle
  \]  
  (3.5)

- More generally, a state is represented by a density operator: a symmetric operator \( \rho \) such that (i) for all \( \psi \in H \), \( \langle \psi, \rho \psi \rangle \geq 0 \) and (ii) \( \text{Tr}(\rho) = 1 \).

- If \( P_q \) is the projector onto the \( q \)-valued eigenspace of \( Q \), then the probability of obtaining the value \( q \) in a \( Q \)-measurement on a system in state \( \rho \) is \( \text{Tr}(\rho P_q) \).

- The expectation value of \( Q \) on the state \( \rho \) is given by
  \[
  \text{Tr} (\rho Q)
  \]  
  (3.6)

- Any pure state (represented by \( \psi \)) is represented by the projector \( P_\psi \) onto the span of \( \psi \) (the subspace \( \{\alpha \psi\}_{\alpha \in \mathbb{C}} \)).

- However, not all density operators take the form of a projector \( P_\psi \) for some \( \psi \): states that are not pure are called mixed states.

- If \( \Xi = \{\xi_1, \ldots, \xi_n\} \) is an orthonormal basis of \( H \), then defining
  \[
  \rho_\Xi(\xi_i) := \langle \xi_i, \rho \xi_i \rangle
  \]  
  (3.7)

  we find that \( \rho_\Xi \) is a discrete probability density on \( \Xi \).

- Thus, any density matrix encodes probability densities over orthonormal bases (and hence, over decompositions into mutually orthogonal subspaces).

- By Gleason’s theorem, the converse is also true for any Hilbert space \( H \) of dimension greater than 2: any assignment \( p \) of probabilities to unit vectors of \( H \), in such a way that the probabilities for any orthonormal basis sum to 1, is expressible in the form
  \[
  p(\psi) = \langle \psi, \rho \psi \rangle
  \]  
  (3.8)

  for some density operator \( \rho \).
4 Algebras of quantities

4.1 *-algebras

- A *-algebra $\mathcal{A}$ is a complex vector space, equipped with the following further structure:
  - A binary operation of multiplication: we denote the product of $X, Y \in \mathcal{A}$ by $XY$
  - A special element $1 \in \mathcal{A}$
  - A unary operation of involution: we denote the involute of $X \in \mathcal{A}$ by $X^*$

- These operations are required to obey the following axioms (in addition to the vector space axioms): for any $X, Y, Z \in \mathcal{A}$ and $\alpha \in \mathbb{C}$,
  \[
  (XY)Z = X(YZ) \quad (4.1)
  
  X1 = X = 1X \quad (4.2)
  
  X(Y + Z) = XY + XZ \quad (4.3)
  
  (X + Y)Z = XZ + YZ \quad (4.4)
  
  X^{**} = X \quad (4.5)
  
  (X + Y)^* = (X^* + Y^*) \quad (4.6)
  
  (\alpha X)^* = \overline{\alpha}X^* \quad (4.7)
  
  (XY)^* = Y^* X^* \quad (4.8)
  
  1^* = 1 \quad (4.9)
  
- A $C^*$-algebra is a *-algebra obeying some further conditions, which we will not discuss: instead, we will confine our attention to finite-dimensional *-algebras, which are guaranteed to be $C^*$-algebras

- A state on $\mathcal{A}$ is a positive, unit-preserving, linear functional over $\mathcal{A}$: that is, a linear map $\omega : \mathcal{A} \to \mathbb{C}$ such that for all $X \in \mathcal{A}$,
  \[
  \omega(X^*X) \geq 0 \quad (\text{hence } \omega(X^*X) \in \mathbb{R}) \quad (4.10)
  
  \omega(1) = 1 \quad (4.11)
  
- An element $X \in \mathcal{A}$ is said to be self-adjoint if $X^* = X$

- If $X$ is self-adjoint, then $\omega(X) \in \mathbb{R}$ for any state $\omega$
4.2 Commuting *-algebras

- Let $W$ be a finite set; then the set $\mathcal{C}(W)$ of all functions $f: W \to \mathbb{C}$ is a *-algebra, where $1$ is the constant function $w \mapsto 1$ and the operations are defined as follows: for any $w \in W$,

\[
(fg)(w) = f(w) \cdot g(w) \quad (4.12)
\]

\[
f^*(w) = \overline{f(w)} \quad (4.13)
\]

- The self-adjoint elements of this *-algebra are the real-valued functions on $W$
- For any probability density $\rho$ on $W$, the expectation-value map $f \mapsto \sum_{w \in W} f(w)p(w)$ is a state
- Moreover, any state $\mathcal{C} \to \mathbb{R}$ is the expectation-value map for some probability density: so states can be identified with probability densities
- Note that $\mathcal{C}$ is commutative: for any $f, g \in \mathcal{C}$, $fg = gf$
- Any finite-dimensional commuting *-algebra can be represented as an algebra of complex functions

4.3 Quantum *-algebras

- Let $H$ be a finite-dimensional Hilbert space; then the set $\mathcal{B}(H)$ of all operators on $H$ is a *-algebra, where $1$ is the identity operator and multiplication is defined as composition
- Involutions are defined via the notion of adjoint: for any operator $O$, its adjoint is the (unique) operator $O^*$ such that for any $\phi, \psi \in H$,

\[
\langle \phi, O^* \psi \rangle = \langle O\phi, \psi \rangle \quad (4.14)
\]

- The self-adjoint elements of this *-algebra are the symmetric operators
- For any density operator $\rho$ over $H$, the expectation-value map $Q \mapsto \text{Tr}(\rho Q)$ is a state
- Moreover, if $\dim(H) \geq 3$, then any state $\mathcal{B}(H) \to \mathbb{R}$ is the expectation-value map for some density operator: so states can be identified with density operators
- Note that such a *-algebra will not, in general, be commutative
- Any finite-dimensional *-algebra can be represented as an algebra of operators (i.e., as a subalgebra\(^1\) of $\mathcal{B}(H)$, for some Hilbert space $H$)

---

\(^1\)A subalgebra of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ which is also a *-algebra: that is, a subspace which contains $1$, and is closed under multiplication and involution.
5 The Kochen-Specker Theorem

5.1 Von Neumann’s no-hidden-variables theorem

- If $\mathcal{E}(Q)$ is the expectation value of a physical quantity $Q$ in some state, then we say that that state is dispersion-free with respect to $Q$ if $\mathcal{E}(Q^2) = \mathcal{E}(Q)^2$

- In classical probability theory, states are probability densities and pure states are characteristic functions

- Hence, classical pure states are dispersion-free with respect to all quantities: they assign a definite value to each such quantity

- For a quantum system represented by a finite-dimensional Hilbert space $H$, physical quantities are represented by symmetric operators on $H$

- The usual states we consider (vectors and density operators) have dispersion with respect to at least some quantities

- Could there be “complete” states which assign definite values to (are dispersion-free with respect to) all quantities?

- An influential negative argument was given by von Neumann in 1935

- First, assume that any state (complete or otherwise) must assign expectation values to quantities in a linear fashion: for any symmetric operators $Q_1$ and $Q_2$, and any $\alpha, \beta \in \mathbb{R}$:

$$\mathcal{E}(\alpha Q_1 + \beta Q_2) = \alpha \mathcal{E}(Q_1) + \beta \mathcal{E}(Q_2)$$  \hspace{1cm} (5.1)

- Second, assume that the definite value any hidden state assigns to a quantity must be an eigenvalue of the corresponding operator

- Then there can be no such hidden states, since (in general) the eigenvalues of a sum of operators are not the sums of the eigenvalues of those operators

- For example, if $S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

then $\frac{1}{\sqrt{2}}(S_x + S_z)$ has eigenvalues $\pm 1$, but $\frac{1}{\sqrt{2}}(\pm 1 \pm 1)$ is either $\sqrt{2}$, 0, or $-\sqrt{2}$.

- However, Bell (and others) have criticised this proof for its requirement that linearity hold even for incompatible (non-commuting) operators, since such operators cannot be simultaneously measured
5.2 The Kochen-Specker theorem

- For the Kochen-Specker theorem, we assume that a hidden-variables state would assign definite (eigen)values to operators in such a way that for compatible (commuting) operators, (5.1) holds.

- It can then be shown that if $H$ has at least three dimensions, no such hidden-variables model is possible.

- For this proof of the theorem, we suppose that $H$ is four-dimensional (which will prove the theorem for four or more dimensions).

- Suppose that $P_1$, $P_2$, $P_3$ and $P_4$ are projectors onto four orthogonal one-dimensional subspaces of $H$.

- Then the $P_i$ are all compatible, and $I = \sum_i P_i$.

- It follows that the hidden-variables state must assign 1 to exactly one of the $P_i$, whilst the other three receive the value 0.

- We then consider the projectors onto the following 18 subspaces, organised into 9 overlapping families of four orthogonal subspaces each:

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- There is no satisfactory way to assign 1s and 0s, provided that we assign the same value to the same projector in two different families (non-contextuality).
6 Joint systems

6.1 Tensor products

- Given a Hilbert space $H$, its dual space $H^*$ is the space of linear maps $\mu : H \to \mathbb{C}$, and is itself a vector space.

- Every $\psi \in H$ may now be regarded as a linear map $H^* \to \mathbb{C}$, given by $\psi(\mu) := \mu(\psi)$.

- The tensor product $H_1 \otimes H_2$ of two Hilbert spaces $H_1$ and $H_2$ consists of all bilinear maps $\Psi : H_1^* \times H_2^* \to \mathbb{C}$: that is, all maps such that (for any $\mu_1, \nu_1 \in H_1^*, \mu_2, \nu_2 \in H_2^*$, and $\alpha, \beta \in \mathbb{C}$):

  \begin{align*}
  \Psi(\alpha \mu_1 + \beta \nu_1, \mu_2) &= \alpha \Psi(\mu_1, \mu_2) + \beta \Psi(\nu_1, \mu_2) \\
  \Psi(\mu_1, \alpha \mu_2 + \beta \nu_2) &= \alpha \Psi(\mu_1, \mu_2) + \beta \Psi(\mu_1, \nu_2)
  \end{align*}

  (6.1)

- $H_1 \otimes H_2$ is also a vector space, with addition and multiplication defined pointwise.

- Any pair $\psi_1 \in H_1$ and $\psi_2 \in H_2$ picks out an element of $H_1 \otimes H_2$ (the tensor product of $\psi_1$ and $\psi_2$), denoted by $\psi_1 \otimes \psi_2$ and given by

  \[(\psi_1 \otimes \psi_1)(\mu_1, \mu_2) = \psi_1(\mu_1) \cdot \psi_2(\mu_2)\]

  (6.3)

- The tensor product commutes with scalar multiplication and distributes over addition: that is, for any $\psi_1, \phi_1 \in H_1, \psi_2, \phi_2 \in H_2$ and $\alpha \in \mathbb{C}$,

  \begin{align*}
  \alpha(\psi_1 \otimes \psi_2) &= (\alpha \psi_1) \otimes \psi_2 = \psi_1 \otimes (\alpha \psi_2) \\
  \psi_1 \otimes (\psi_2 + \phi_2) &= (\psi_1 \otimes \psi_2) + (\psi_1 \otimes \phi_2) \\
  (\psi_1 + \phi_1) \otimes \psi_2 &= (\psi_1 \otimes \psi_2) + (\phi_1 \otimes \psi_2)
  \end{align*}

  (6.4) (6.5) (6.6)

- However, not all elements of $H_1 \otimes H_2$ are expressible as a tensor product: for instance, $(\psi_1 \otimes \psi_2) + (\phi_1 \otimes \phi_2)$ is not (in general) expressible this way.

- That said, all elements of $H_1 \otimes H_2$ are expressible in terms of sums of pairs: if $\{\delta_1^i\}_i$ is a basis for $H_1$, and $\{\delta_2^j\}_j$ is a basis for $H_2$, then $\{\delta_1^i \otimes \delta_2^j\}_{i,j}$ is a basis for $H_1 \otimes H_2$.

- For any $\Psi = \sum_{ij} \Psi_{ij} (\delta_1^i \otimes \delta_2^j)$ and $\Phi = \sum_{kl} \Phi_{kl} (\delta_1^k \otimes \delta_2^l)$, the inner product is

  \[\langle \Psi, \Phi \rangle = \sum_{ijkl} \Psi_{ij} \Phi_{kl} (\delta_1^i, \delta_1^k)(\delta_2^j, \delta_2^l)\]

  (6.7)
6.2 Tensor space operators

- Given any linear operators $O_1 : H_1 \rightarrow H_1$ and $O_2 : H_2 \rightarrow H_2$, there is an associated linear operator $O_1 \otimes O_2 : H_1 \otimes H_2 \rightarrow H_1 \otimes H_2$

- We define it as follows: given an arbitrary $\Psi = \sum_{ij} \Psi_{ij} (\delta_i^1 \otimes \delta_j^2)$,

$$O_1 \otimes O_2(\Psi) = \sum_{ij} \Psi_{ij} \left( O_1(\delta_i^1) \otimes O_2(\delta_j^2) \right)$$  (6.8)

- Again, not all linear operators on $H_1 \otimes H_2$ correspond to such pairs, but they are all expressible as sums such of pairs

- An operator $O_1$ acting only on $H_1$ can be represented by an operator on $H_1 \otimes H_2$ of the form $O_1 \otimes I_2$, where $I_2$ is the identity operator on $H_2$

6.3 Entanglement

- If we have two quantum systems, represented by Hilbert spaces $H_1$ and $H_2$ respectively, then we may regard the pair of systems together as a joint system with Hilbert space $H_1 \otimes H_2$

- The observables for just one system are always compatible with those for the other, since (for any symmetric $Q_1$ and $Q_2$) $Q_1 \otimes I_2$ will commute with $I_1 \otimes Q_2$

- States which are not expressible in the form $\psi_1 \otimes \psi_2$ are said to be entangled

- For example, suppose we are working with a pair of simple spin systems, in the z-spin basis for both, and consider the entangled state

$$\Psi = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$  (6.9)

- Although this is a pure state of the joint system, the individual systems are not in pure states: instead, they are in the mixed states

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$  (6.10)

- However, these mixed states do not uniquely determine the joint state, since they do not encode the correlations between the different outcomes

- For instance, if we measure the z-spin of both systems (i.e. the quantity represented by $S_z \otimes S_z$), then we are guaranteed to get +1 on one system and -1 on the other: so by measuring the z-spin on one, we can predict the z-spin of the other with certainty (note that $S_z \otimes I$ and $I \otimes S_z$ commute)
7 Bell’s theorem I

7.1 Conditional probabilities

- Let $\rho$ be a probability density over (finite) sample space $W$
- Recall that $\rho$ defines a probability $p(E)$ for each even $E \subseteq W$
- For any two events $E, F$, the probability of $E$ conditional on $F$ is defined as

$$p(E|F) := \frac{p(E \cap F)}{p(F)} \quad (7.1)$$

- $E$ and $F$ are said to be independent when

$$p(E \cap F) = p(E)p(F) \quad (7.2)$$

which is equivalent to the condition that

$$p(E|F) = p(E) \quad (7.3)$$

- For compactness, we will write $p(E, F)$ for $p(E \cap F)$ and $p(E|F_1, F_2)$ for $p(E|F_1 \cap F_2)$
- Note that if $F_1, \ldots, F_n$ are mutually exclusive and jointly exhaustive events, then

$$p(E) = p(E, F_1) + \cdots + p(E, F_n) \quad (7.4)$$

$$= p(E|F_1)p(F_1) + \cdots + p(E|F_n)p(F_n) \quad (7.5)$$

7.2 Alice and Bob

- Our setups will concern a pair of experimenters, Alice and Bob, who perform measurements on a pair of (previously interacting, but now separated) systems
- All their measurements have only two outcomes, “Y” (+1) and “N” (-1)
- We will be interested in the probabilities for Alice and Bob to get certain outcomes, conditional on their having performed certain measurements
- The probability that Alice gets the outcome “Y”, conditioned on Alice performing the measurement $A$ and Bob performing the measurement $B$, will be denoted

$$p(Y^a|A^a, B^a) \quad (7.6)$$
7.3 Correlation arrays

- We will use correlation arrays (due to Bub) to display these conditional probabilities.

- Suppose that Alice is choosing among \( n \) possible measurements \( A_1, \ldots, A_n \) and Bob is choosing among \( p \) possible measurements \( B_1, \ldots, B_p \); then the associated correlation array consists of an \( n \times p \) array of \( 2 \times 2 \) “cells”, where each cell displays the probabilities conditional on a given pair of measurements being performed.

- Thus, if Alice is choosing between performing the measurements \( Q \) and \( P \), and Bob is choosing between the measurements \( R \) and \( S \), then the associated correlation array looks like this:

<table>
<thead>
<tr>
<th></th>
<th>( Y )</th>
<th>( N )</th>
<th>( Y )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( Y )</td>
<td>( p(Y^a, Y^b</td>
<td>Q^a, R^b) )</td>
<td>( p(N^a, Y^b</td>
</tr>
<tr>
<td>( N )</td>
<td>( p(Y^a, N^b</td>
<td>Q^a, R^b) )</td>
<td>( p(N^a, N^b</td>
<td>Q^a, R^b) )</td>
</tr>
<tr>
<td>( S )</td>
<td>( Y )</td>
<td>( p(Y^a, Y^b</td>
<td>Q^a, S^b) )</td>
<td>( p(N^a, Y^b</td>
</tr>
<tr>
<td>( N )</td>
<td>( p(Y^a, N^b</td>
<td>Q^a, S^b) )</td>
<td>( p(N^a, N^b</td>
<td>Q^a, S^b) )</td>
</tr>
</tbody>
</table>

- Since each \( 2 \times 2 \) cell describes a conditional probability distribution, the values in each cell must sum to 1.

- For example, suppose that Alice and Bob carry a pair of fair coins, and that the measurement consists of choosing a coin and tossing it (with heads as “Y” and tails as “N”); then the resulting correlation array is:

<table>
<thead>
<tr>
<th></th>
<th>( Y )</th>
<th>( N )</th>
<th>( Y )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Bob's 1st coin} )</td>
<td>( Y )</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>( \text{Bob's 2nd coin} )</td>
<td>( N )</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

- Note that one cannot infer the (unconditional) probabilities from the information in a correlation array, without knowing the probabilities for Alice and Bob to make particular measurements.
8 Bell’s theorem II

8.1 Some facts about spin

- We have already met, and made extensive use of, the z-spin and x-spin operators:

\[
S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (8.1)

- More generally, the operator associated with a spin measurement at angle \( \theta \) from the z-axis, in the x-z plane, is given (in the z-spin basis) by

\[
S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}
\] (8.2)

- The singlet state is given by

\[
\Psi = \frac{1}{\sqrt{2}} (\uparrow \otimes \downarrow - \downarrow \otimes \uparrow)
\] (8.3)

where \( \uparrow \) and \( \downarrow \) are the eigenstates of z-spin

- Remarkably, the singlet state takes this form relative to the basis of spin eigenstates for any axis

- This means that if we Alice and Bob are working with a singlet state, then if they measure spin along the same axis (on their respective systems), they will always get opposite results

- More generally, if they measure along axes separated by an angle \( \phi \), then the probability that they get different results is given by

\[
P_\phi = \frac{1 + \cos \phi}{2}
\] (8.4)
8.2 Bell’s inequality

• Let $\lambda \in \Lambda$ be the (hypothetical) “complete state” of Alice and Bob’s systems
• Let $a$ denote Alice’s measurement, and $b$ denote Bob’s measurement, each of which is given by a direction in the $x$-$z$ plane
• Let $s$ denote the outcome of Alice’s measurement, and let $t$ denote the outcome of Bob’s measurement: we suppose that the possible outcomes of any measurement are $\pm 1$
• In Bell’s original proof, he supposed that the value of $s$ was determined by Alice’s choice of measurement $a$ together with the complete state $\lambda$, and similarly for $t$
• This amounts to treating $\Lambda$ as a sample space, equipped with some probability density $\rho(\lambda)$, and considering families of random variables $s_a$ and $t_b$:

\[
\begin{align*}
s &= s_a(\lambda) \\
t &= t_b(\lambda)
\end{align*}
\]

• Now consider any three directions $k, l, m$ as possible measurements for Alice: since $s$ can only take two values, the three events

\[
\begin{align*}
E &:= \{ \lambda : s_k(\lambda) = s_l(\lambda) \} \\
F &:= \{ \lambda : s_l(\lambda) = s_m(\lambda) \} \\
G &:= \{ \lambda : s_m(\lambda) = s_k(\lambda) \}
\end{align*}
\]

must cover the whole of $\Lambda$
• It follows that the probabilities of these events must sum to at least 1:

\[
p(E) + p(F) + p(G) \geq 1
\]

• We know that when $a = b$, Alice and Bob’s outcomes are perfectly anticorrelated: thus $s_k(\lambda) = -t_k(\lambda)$, and similarly for $l$ and $m$
• It follows that $E = \{ \lambda : s_k(\lambda) \neq t_l(\lambda) \}$, and similarly for $F$ and $G$
• Hence,

\[
p(s_k \neq t_l) + p(s_l \neq t_m) + p(s_m \neq t_k) \geq 1
\]

• But if we now choose $k, l$ and $m$ so as to be separated by $120^\circ$, then we find that

\[
p(s_k \neq t_l) = \frac{1 + \cos 120^\circ}{2} = \frac{1}{4}
\]

and the same for the others; so we have a contradiction.
8.3 The CHSH inequality

• The above derivation assumed that the complete state fully determined the results of all measurements, and that Alice and Bob’s outcomes were perfectly correlated.

• However, we can relax these assumptions and still obtain an inequality (due to Clauser, Horne, Shimony and Holt) which is violated by quantum mechanics.

• Thus, we will suppose that there is merely a probabilistic dependence of Alice and Bob’s outcomes $s$ and $t$ on the hidden state $\lambda$ (and on their choices of measurement $a$ and $b$).

• We now make the following assumptions:
  
  – **No conspiracy:**
    \[ P(\lambda|a, b) = P(\lambda) \] (8.13)
  
  – **Outcome independence:**
    \[
    P(s|\lambda, a, b, t) = P(s|\lambda, a, b) \] (8.14a)
    \[
    P(t|\lambda, a, b, s) = P(t|\lambda, a, b) \] (8.14b)
  
  – **Parameter independence:**
    \[
    P(s|\lambda, a, b) = P(s|\lambda, a) \] (8.15a)
    \[
    P(t|\lambda, a, b) = P(t|\lambda, b) \] (8.15b)

• The conjunction of (8.14) and (8.15) is equivalent to the assumption of Bell locality:
  \[
  P(s, t|\lambda, a, b) = P(s|\lambda, a)P(t|\lambda, b) \] (8.16)

• Let $E(a, b)$ be the (unconditional) expectation value of the product $st$ of measuring $a$ and $b$:
  \[
  E(a, b) := \sum_s \sum_t (st) \cdot P(s, t|a, b) \] (8.17)

• By combining (8.16), (8.17), and some general facts about expectation values, we obtain the CHSH inequality: for any measurements $a, a'$ by Alice and $b, b'$ by Bob,
  \[
  -2 \leq E(a, b) + E(a', b) + E(a, b') - E(a', b') \leq 2 \] (8.18)
Violation of the CHSH inequality

- Assume that we have fixed a coordinate system $x, y, z$; let the $k$- and $m$-axes be as depicted (i.e. to be the result of rotating the $x$- and $z$-axes by $45^\circ$ in the $(x, z)$ plane):

- Now suppose that we choose the following four measurements:
  - $a$ is measurement of $z$-spin (of Alice’s system)
  - $a'$ is measurement of $x$-spin (of Alice’s system)
  - $b$ is measurement of $k$-spin (of Bob’s system)
  - $b'$ is measurement of $m$-spin (of Bob’s system)

- Now suppose that we are working with the singlet state $\Psi$: if $a$ and $b$ are spin along two axes separated by an angle $\phi$, then the expectation value is

$$E(a, b) = P_\phi(-1) + (1 - P_\phi)(1) = -\cos \phi$$

(8.19)

- It follows that the expectation values are:

$$E(a, b) = E(a', b) = E(a, b') = -\frac{1}{\sqrt{2}}\quad E(a', b') = \frac{1}{\sqrt{2}}$$

(8.20)

and hence that

$$E(a, b) + E(a', b) + E(a, b') - E(a', b') = -2\sqrt{2} < -2$$

(8.21)

- Thus, quantum mechanics violates the CHSH inequality, which we derived as a general condition on any complete, local theory

- Thus, there is no complete, local theory which replicates the empirical predictions of quantum mechanics.

- These empirical predictions have been borne out experimentally: so (it would appear) there cannot be a complete, local theory which is empirically adequate.
9 The measurement problem

9.1 Interference

- Suppose that beam of x-spin-up electrons is fed into a pair of Stern-Gerlach apparatuses, measuring first z-spin and then x-spin.

- From what we already know, we can predict the probability of getting various results:

  \[
  \begin{array}{c|c|c}
  \text{Source (u)} & \text{Z} & \text{X} \\
  \hline
  +1 & 25\% & +1 \\
  -1 & 25\% & +1 \\
  +1 & 25\% & -1 \\
  -1 & 25\% & -1 \\
  \end{array}
  \]

- However, what if we use a pair of reflecting mirrors to recombine the output beams from the first apparatus, and feed that into a single x-spin Stern-Gerlach apparatus?

- If we do this, then we obtain the following results:

  \[
  \begin{array}{c|c|c}
  \text{Source (u)} & \text{Z} & \text{X} \\
  \hline
  +1 & - & +1 \\
  -1 & 100\% & -1 \\
  +1 & - & 0\% \\
  -1 & - & - \\
  \end{array}
  \]

- It is tempting to suppose that the electrons are somehow “jostling” each other when the streams are recombined; but the effect is still present if we only send one electron through at a time.

- This phenomenon is known as interference: it shows that so long as a system is not being measured, quantum states evolve linearly.

- Note that if we insert a detector along one path, so that we know which path the electrons are taking, then the x-spin results are 50–50!
9.2 Schrödinger’s cat

• Thus, at least in non-measurement contexts, dynamical evolution is represented by linear operators on the Hilbert space \( H \): for a given stretch of time \( \Delta t \), there is an operator \( U_{\Delta t} \) such that for any state \( \psi \in H \), \( U_{\Delta t} \psi \) is what \( \psi \) has evolved into after time \( \Delta t \).

• Being linear, these operators commute with superpositions: that is,

\[
U_{\Delta t}(c_1\psi + c_2\phi) = c_1U_{\Delta t}\psi + c_2U_{\Delta t}\phi
\]  

(9.1)

• Suppose we quantum-mechanically model Schrödinger’s cat: an electron is fed through a Stern-Gerlach apparatus, then sounds a bell if it emerges from the “z-spin-up” exit or releases poison if it emerges from the “z-spin-down” exit.

• If \( N \) is the state of the cat napping, \( L \) is the state of the cat being alive (and awake), and \( D \) is the state of the cat being dead, then the evolutions of the joint electron-cat system for z-spin eigenstates are clearly

\[
\uparrow \otimes N \to \uparrow \otimes L
\]

(9.2a)

\[
\downarrow \otimes N \to \downarrow \otimes D
\]

(9.2b)

• It follows that the evolution for an x-spin-up electron should result in the state

\[
\frac{1}{\sqrt{2}}(\uparrow \otimes L + \downarrow \otimes D)
\]

(9.3)

• But the standard quantum-mechanical formalism says that the outcome of a measurement is not (9.3), but rather

\[
either \uparrow \otimes L
\]

(9.4a)

\[
or \downarrow \otimes D
\]

(9.4b)

with a 50% probability of each.

• This is (in brief form) the measurement problem: the standard formalism seems to give different answers to the question “what is the final state?”, depending on whether a “measurement” has taken place or not.
9.3 Curiosity about the cat

- (9.3) and (9.4) are distinct states, which are (in principle) empirically distinguishable
- Let the cat’s “undeadness” be a quantity with vampire and zombie eigenstates

\[
V := \frac{1}{\sqrt{2}}(L + D) \quad (9.5a)
\]
\[
Z := \frac{1}{\sqrt{2}}(L - D) \quad (9.5b)
\]

- Undeadness is (in principle) measurable, but this won’t suffice to distinguish (9.3) from (9.4), since all three of the states (9.3), (9.4a), and (9.4b) are associated with equal probabilities of finding vampire or zombie
- One can show, in fact, that there is no property of the electron or cat individually which will distinguish (9.4) from (9.3)
- However, if we measure both x-spin and undeadness, then we are able to distinguish between (9.3) and (9.4): either state in (9.4) has a non-zero chance of yielding outcomes “x-spin-down and vampire” or “x-spin-up and zombie”, whereas (9.3) assigns zero probability to both
- If the states go on to further interact with their environment, then even this will fail
- Suppose that there is a bowl of food in the box, which the cat will eat if it survives
- Then we are trying to determine whether the post-measurement state is

\[
\frac{1}{\sqrt{2}}(\uparrow \otimes L \otimes E + \downarrow \otimes D \otimes F) \quad (9.6)
\]

where \(E\) is the state of the bowl being empty and \(F\) is the state of the bowl being full, or whether it is

- either \(\uparrow \otimes L \otimes E\) \quad (9.7a)
- or \(\downarrow \otimes D \otimes F\) \quad (9.7b)

with 50% probability for each
- (9.6) and (9.7) yield the same probabilities for joint x-spin-and-undeadness measurements; we can only distinguish them by a joint measurement of electron, cat and bowl
9.4 The measurement problem(s)

- Exactly what the measurement problem is and how it ought to be stated is itself controversial (in part because different versions of the problem tend to suggest different solutions)

- Albert (1992)'s statement stresses the idea that the state (9.3) is intrinsically bizarre: this suggests classifying solutions according to whether they deny that such states arise, or whether they somehow explain why these states are not so bizarre as they seem

- Maudlin (1995)'s first version states that the problem is the joint inconsistency of the following three statements:
  1. The quantum state $\psi$ of a system is complete
  2. The quantum state always evolves in accordance with linear Schrödinger dynamics
  3. Measurements always (or at least usually) have determinate outcomes

- This taxonomises responses to the measurement problem as follows:
  - Responses which deny 1 are hidden-variable (or better, “additional-variable”) theories
  - Responses which deny 2 are collapse (or better, “non-linear”) theories
  - Responses which deny 3 are multiverse (or nulliverse) theories

- The second version also points to an inconsistency, this time between 1, 2 and 3.’ Measurement situations with identical initial states sometimes have different outcomes, whose probabilities are given by the Born rule

- This version (claims Maudlin) rules out multiverse theories, since they are unable to account adequately for the Born rule

- Wallace (2012): sometimes, we think of elements of a state space as representing physically distinct configurations of the system under investigation; and sometimes, we think of them as representing probability distributions over such configurations

- When dealing with macroscopic quantum states (e.g. (9.3)) we tend to employ the probabilistic conception, and take such a state represents a probability distribution over different measurement outcomes (each with probability 50%)

- When dealing with microscopic quantum states, however, we must employ the physical conception (on pain of failing to account for interference phenomena)

- The measurement problem, for Wallace, is that we cannot consistently apply one conception or the other, and that the transition between the two seems to be delimited only in a vague, rough-and-ready sort of way
10 Bohmian mechanics

10.1 Wavefunctions

• The state of a single (spinless) particle is given by a wavefunction: a map

\[ \psi : T \times \Sigma \to \mathbb{C} \]  \hspace{1cm} (10.1)

where \( T \) and \( \Sigma \) represent time and space

• Given a particle with wavefunction \( \psi \), if a measurement is performed at time \( t \) to determine if the particle is in some region \( \Delta \subseteq \Sigma \), then the probability of getting a positive answer is

\[ \int_{x \in \Delta} |\psi(t, x)|^2 \, dx \]  \hspace{1cm} (10.2)

• In other words, the wavefunction defines a probability density

\[ \rho(t, x) = |\psi(t, x)|^2 \]  \hspace{1cm} (10.3)

• Like any probability density, this is required to be normalised: for any time \( t \),

\[ \int_{x \in \Sigma} |\psi(t, x)|^2 \, dx = 1 \]  \hspace{1cm} (10.4)

• On standard approaches to QM, if we measure a particle as being in the region \( \Delta \), then immediately after the measurement its wavefunction is given by the (normalised) restriction of \( \psi \) to \( \Delta \)

• Outside measurement contexts, the wavefunction evolves according to Schrödinger’s equation:

\[ \frac{d\psi}{dt} = H\psi \]  \hspace{1cm} (10.5)

where \( H \) is the Hamiltonian

• For example, the Hamiltonian for a particle of mass \( m \) in a potential described by a function \( V : \Sigma \to \mathbb{R} \) is

\[ H = \frac{1}{2m} \nabla^2 + V \]  \hspace{1cm} (10.6)

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \)
10.2 The guidance equation

- The fundamental entities of Bohmian mechanics are the particles: pointlike objects, which have definite positions at all times, and which are held to be the fundamental constituents of macroscopic matter.

- Given a single particle of mass $m$, its position $Q \in \Sigma$ evolves deterministically over time in a manner determined by the wavefunction $\psi : \Sigma \to \mathbb{C}$, as described by the guidance equation:

$$\frac{dQ}{dt} = \frac{\hbar}{m} \text{Im} \left( \frac{\nabla \psi}{\psi}(Q) \right)$$

(10.7)

- Since the guidance equation is first-order, the wavefunction determines the particle’s velocity.

- The evolution of the particle depends on the evolution of the wavefunction, but not vice versa: the wavefunction always evolves according to the usual Schrödinger equation (10.5), even in measurement contexts.

- Note that the probability density $|\psi(x)|^2$ is equivariant: evolving the wavefunction via (10.5) delivers the same result as evolving the probability density via (10.7).

- In each run of the two-slit experiment the particle goes through one slit or another, but the interference within the wavefunction “guides” the particle towards certain parts of the screen and away from others (see Figure 10.1).

![Bohmian trajectories for the two-slit experiment.](image)

- The outcome of a non-position measurement depends, in general, on the method of measurement used: this provides a sense in which the theory is contextual.
10.3 The notion of primitive ontology

Bell, *The Theory of Local Beables* (1975):

The concept of ‘observable’ lends itself to very precise mathematics when identified with ‘self-adjoint operator’. But physically, it is a rather woolly concept. [...] So it could be hoped that some increase in precision might be possible by concentration on the beables, which can be described in classical terms, because they are there.


[...] in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the ‘measurement’ of anything else, then you commit redundancy and risk inconsistency.


According to (pre-quantum-mechanical) scientific precedent, when new mathematically abstract theoretical entities are introduced into a theory, the physical significance of these entities, their very meaning insofar as physics is concerned, arises from their dynamical role, from the role they play in (governing) the evolution of the more primitive—more familiar and less abstract—entities or dynamical variables. [...] Why would these abstractions be introduced in the first place, if not for their relevance to the behavior of something else, which somehow already has physical significance?

10.4 Several particles

- If we have $N$ particles, then their collective state at any given time may be represented by an $N$-tuple $(Q_1, \ldots, Q_N) \in \Sigma^N$, i.e. by a point in configuration space.

- This time, the guidance equation describes the evolution of the collection of particles, as determined by the wavefunction $\Psi : \Sigma^N \to \mathbb{C}$

\[
\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \text{Im} \left( \frac{\nabla_i \Psi}{\Psi} (Q_1, \ldots, Q_N) \right)
\]  

(10.8)

where $m_i$ is the mass of the $i$th particle, and $\nabla_i$ denotes the position-derivative associated with the $i$th factor of $\Sigma^N$.

- Note that this dynamics is non-local, since the wavefunction is a function on configuration space (and the motion of any one particle depends on the wavefunction’s value at the “occupied” point of configuration space).
10.5 Subsystems

- Suppose that we have some system of \( N \) particles governed by Bohmian mechanics, as described above.
- Now say the first \( M < N \) particles are a subsystem, with the remaining particles being the environment.
- Let \( x \) be coordinates for \( \Sigma^M \) and \( y \) be coordinates for \( \Sigma^{N-M} \), and write \( Q = (X, Y) \).
- Given a particular environment configuration \( Y \in \Sigma^{N-M} \), define the conditional wavefunction of the subsystem as
  \[ \psi(x) := \Psi(x, Y) \tag{10.9} \]
  The conditional wavefunction determines the conditional probability distribution for the subsystem configuration (conditional, that is, on the environment configuration):
  \[ \rho(x|Y) = |\psi(x)|^2 \tag{10.10} \]
- Note that the conditional wavefunctions of subsystem and environment underdetermine the full system wavefunction \( \Psi \).
- Now suppose that the total-system wavefunction is of the form
  \[ \Psi(x, y) = \chi(x)\phi(y) + \Psi_\perp(x, y) \tag{10.11} \]
  where \( \Psi_\perp \) and \( \phi \) have disjoint support in the environment coordinates \( y \).
- If \( Y \in \text{supp}(\phi) \), then the guidance equation for \( X \) reduces to
  \[ \frac{dX_i}{dt} = i\frac{h}{m_i} \text{Im} \left( \nabla_i \chi(X) \right) \tag{10.12} \]
  and we refer to \( \chi \) as the effective wavefunction for the subsystem.
- Moreover, observe that if \( Y \in \text{supp}(\phi) \), then \( \chi = \psi \) (i.e., the effective wavefunction coincides with the conditional wavefunction).
- Replacing the total-system wavefunction by its conditional/effective wavefunction is referred to as a process of “effective collapse”; this is what justifies the (informal) use of the collapse postulate by Bohmians.
- It is in this sense that one can make use of Bohmian mechanics, even in the absence of data about the universal wavefunction.
- It also means that effective wavefunctions may be very different from the universal wavefunction: one could have time-dependent effective wavefunctions yet a time-independent universal wavefunction, for example.
11 Symmetry in quantum mechanics

11.1 SO(3) and spin

- The group SO(3) consists of all rotations (length-, angle- and orientation-preserving transformations) in three-dimensional space.
- A matrix $M$ is said to be orthogonal if its transpose is its inverse, i.e. if $\sum_i M_{ij}M_{ik} = I_{jk}$, where $I_{jk}$ is the identity matrix.
- Orthogonal $3 \times 3$ matrices preserve the standard inner product on $\mathbb{R}^3$, and therefore represent transformations that preserve lengths and angles.
- The determinant of a $3 \times 3$ matrix represents the change in signed volume of the unit cube when the matrix is applied to $\mathbb{R}^3$.
- For an orthogonal matrix, the determinant must be $+1$ or $-1$; if the determinant is $+1$, then the orthogonal matrix is said to be special.
- Hence, the rotation group consists of all special orthogonal matrices.
- The Pauli matrices are the three complex $2 \times 2$ matrices (i.e. operators on $\mathbb{C}^2$)

$$
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
$$

(11.1)

- Writing these as a $1 \times 3$ column vector $\boldsymbol{\sigma}$, then the spin operator in an arbitrary direction (unit vector) $\hat{n}$ is given by

$$
S_{\hat{n}} = \frac{\hbar}{2} \hat{n} \cdot \boldsymbol{\sigma}
$$

(11.2)

- This operator represents the quantity of spin (in units of angular momentum) measured by a Stern-Gerlach apparatus aligned along the $\hat{n}$-axis.
- For any rotation $R \in SO(3)$, there is an associated unitary operator $U_R : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that

$$
U_RS_{\hat{n}}U_R^{-1} = S_{R\hat{n}}
$$

(11.3)

- This means that if $\psi$ is an eigenvector of $S_{\hat{n}}$, then $U_R\psi$ is an eigenvector (with corresponding eigenvalue) of $S_{R\hat{n}}$.  

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11.2 Permutation operators

- Given a set of $N$ objects, the \textit{symmetric group} $\text{Sym}(N)$ consists of all permutations of that set.

- If we have a collection of $N$ indistinguishable quantum particles, then we anticipate that permutations of these particles will be symmetries of the total system with joint Hilbert space

$$H = H_1 \otimes \cdots \otimes H_N$$

where $H_i = H_j$, for each $i, j$.

- Hence, for every permutation $p : \{1, \ldots, N\} \to \{1, \ldots, N\}$, there should be an associated operator $P : H \to H$ with the property that for any $\psi_i \in H_i$,

$$P(\psi_1 \otimes \cdots \otimes \psi_N) = \psi_{p(1)} \otimes \cdots \otimes \psi_{p(N)}$$

- The \textit{indistinguishability postulate} demands that physically admissible states be such that the expectation values of all quantities are invariant under permutations of indistinguishable particles: that is, it requires that for any physical quantity $Q$ and permutation $p$, 

$$\langle \psi, Q\psi \rangle = \langle P\psi, QP\psi \rangle$$

- We say that a pure (joint) state $\Psi \in H$ is \textit{symmetric} if, for any permutation $p$,

$$P\Psi = \Psi$$

and \textit{anti-symmetric} if for any permutation $p$,

$$P\Psi = \text{sgn}(p)\Psi$$

- The \textit{symmetrisation postulate} requires that physically admissible states be either symmetric or anti-symmetric: it entails the indistinguishability postulate, but is not entailed by it.

- The symmetric and anti-symmetric vectors in $H$ form two mutually orthogonal subspaces.

- In the event that $N = 2$, these two subspaces span $H$: that is, any vector can be expressed as a sum of symmetric and anti-symmetric states.

- Indeed, if the $\psi_i$ form a basis for $H_1 = H_2$, then a basis for $H$ is given by states of the form

$$\psi_i \otimes \psi_j + \psi_j \otimes \psi_i, \quad \psi_i \otimes \psi_j - \psi_j \otimes \psi_i$$

- However, if $N > 2$, then $H$ is not spanned by the symmetric and anti-symmetric vectors: the states lying outside this span are known as \textit{paraparticle} states.
12 The Everett interpretation

12.1 Everett’s “relative state” interpretation

- Consider again the state obtained by using linear quantum dynamics to model (our variant on) Schrödinger’s cat:

\[
\frac{1}{\sqrt{2}} (\uparrow \otimes L + \downarrow \otimes D)
\] (12.1)

- The essence of the Everett interpretation is the suggestion that we not fear states like (12.1); instead,

The general validity of pure wave mechanics, without any statistical assertions, is assumed for all physical systems, including observers and measuring apparata. Observation processes are to be described completely by the state function of the composite system which includes the observer and his object-system, and which at all times obeys the wave equation. (Everett 1973, p. 8)

- However, this appears to simply mean biting the bullet: as the old saw goes, surely a cat cannot be both alive and dead?

- In Everett’s original formulation, the suggestion is that although a state like (12.1) does not provide (realistic) states for the electron and cat, it does provide them with relative states: the cat is in the state \(L\) relative to the electron being in the state \(\uparrow\), and it is in the state \(D\) relative to the electron being in the state \(\downarrow\).

- Similarly, Wigner’s friend has different belief-states relative to the different possible laboratory outcomes

- A common interpretation (albeit one that was not used so much by Everett) is that the terms in (12.1) represent how things stand according to different worlds

- To say that the cat’s state is \(L\) relative to the electron’s state being \(\uparrow\) means that the world in which the electron is measured as spin-up is a world in which the cat is alive
12.2 The preferred basis problem and decoherence

- Given the definitions introduced in session 9, the state (12.1) can equally well be written as
  \[
  \frac{1}{\sqrt{2}} (\uparrow_x \otimes V + \downarrow_x \otimes Z)
  \]  
  (12.2)

- So are we dealing with a pair of worlds with an electron in eigenstates of z-spin and a cat in eigenstates of aliveness, or a pair of worlds with an electron in eigenstates of x-spin and a cat in eigenstates of undeadness?

- In session 9, we saw how a system entangled with its environment can have interference terms suppressed

- This occurs when the environmental states associated with the system’s eigenstates (for a particular quantity) are reasonably orthogonal, i.e. can themselves be regarded as reasonably distinct eigenstates of some property of the environment

- Speaking very roughly, decoherence is the suppression of interference over time by entanglement with an environment capable of recording historical data

- That is, if \( q := (q_1, \ldots, q_N) \) is a temporal sequence of eigenstates of some system’s quantity \( Q \), and \( \phi(q) \) is the state that the environment would be in at the end of that sequence, then we require that \( \phi(q) \) and \( \phi(q') \) are distinct eigenstates of some environmental quantity (unless \( q = q' \))

- Note that this is nontrivial: a priori, an environment might fail to record historical data at all (think of footprints on a seashore), or it might fail to record chronology (think of a collection of bullet holes)

- In general, it is a special feature of a quantity \( Q \) (of the system) that it is associated with decoherence in this fashion: for example, the cat’s environment will decohere the cat with respect to liveliness but not with respect to undeadness

- As we saw before, the synchronic suppression of interference between eigenstates of \( Q \) means that a system’s state has the mathematical structure of a probabilistic mixture of \( Q \)-determinate states

- The diachronic suppression of interference, between distinct histories of eigenstates of \( Q \), means that the system’s evolution has the mathematical structure of a probabilistic mixture of quasi-classically evolving \( Q \)-determinate histories

- More carefully, it is a probabilistic mixture of such histories which are distinct at this point in time; as time goes on, there are more opportunities for histories to diverge

- In this sense, the quantum dynamics exhibits a branching structure (the “splitting of worlds”)
12.3 The problem of uncertainty

• How does it even make sense to talk of probabilities, if every outcome actually occurs?

• To answer this question, we must consider what account(s) of ontology one could give within the Everett interpretation

• So suppose that some measurement has been performed, and witnessed by an experimenter, giving rise (via decoherence) to a branching dynamics of the kind discussed above

• How many physical objects (experimenter bodies) are there after the branching event?

  1. If the answer is 2 or more: how many physical objects were there before the branching event?
     a) If the answer is 2 or more, then we have a physical divergence account (cf. Sebens 2015)
     b) If the answer is 1, then we have a physical fission account (cf. DeWitt 1970)

  2. If the answer is 1: how many experimenter minds are there after the branching event?
     a) If the answer is 2 or more: how many experimenter minds were there before the branching event?
        i. If the answer is 2 or more, then we have a mental divergence account (e.g. the “many-minds view” in Albert and Loewer 1988)
        ii. If the answer is 1, then we have a mental fission account (cf., possibly, Saunders 1998)
     b) If the answer is 1: is that mind always in a determinate mental state?
        i. If the answer is “yes”, then we have an anti-psychophysical-parallelism account (e.g. the “single-mind view” in Albert and Loewer 1988)
        ii. If the answer is “no”, then we have a bare theory account (e.g. that discussed in Albert 1992)

• On accounts with multiple agents post-branching, or on Albert and Loewer’s single-mind view, the probabilities can be understood as indicative of an agent’s subjective uncertainty prior to the measurement (about which agent one will become, about which agent one currently is, or about what will happen to oneself); the former can also seek to understand them as measures of interest

• There seems to be a consensus that the bare theory is unable to offer a satisfactory answer to the problem of incoherence; for this (and other reasons) it is not a popular account of ontology for the Everett interpretation
12.4 The problem of probability

- Even if we grant that using probabilities is appropriate in the context of the Everett interpretation, what justifies the claim that the probabilities to be used are those given by textbook quantum mechanics?

- An immediately tempting answer: the probability of an outcome is the proportion of branches where that outcome occurs

- However, branches are not discrete entities (on the decoherence-based approach), so we cannot determine the proportion by simple counting

- Moreover, insofar as we do have means of individuating branches it seems that the branch-counting method would often give the wrong result: for example, it would assign 50-50 chances to a z-spin measurement of an electron in the state

\[
\frac{1}{2} \uparrow + \frac{\sqrt{3}}{2} \downarrow
\]  

(12.3)

and it gives inconsistent answers if applied to multiple branching events

- An alternative is to refine this idea, by claiming that the Born-rule weight of an eigenstate gives the proportion of branches with the outcome corresponding to that eigenstate: but the question is then what justifies that claim

- A more recent approach is to exploit the tools of decision theory

  - In classical decision theory, one can show that a rational agent, confronted by the need to make a choice of bets with uncertain outcomes, will behave as though they are maximising expected utility relative to some utility function and probability distribution over outcomes

  - The aim for the Everettian decision theorist is to show that a rational agent, confronted by a choice of “quantum bets” (bets on the outcomes of quantum measurements), will behave as though they are maximising expected utility relative to some utility function and the Born rule probability distribution over outcomes

  - The extra ingredient, which enables the deduction of a specific probability distribution, is the so-called assumption of “measurement neutrality”:

    A rational agent is indifferent between any two quantum bets which agree on: (i) the state \( \psi \) to be measured; (ii) the physical quantity \( Q \) to be measured; and (iii) the payoff function from outcomes of the measurement to rewards (or penalties)

- However, it remains controversial whether the premises of the proof are appropriate, and over whether this decision-theoretic strategy succeeds in recovering a sufficiently robust notion of probability
13 Quantum Bayesianism

13.1 QBism

- QBism (previously Quantum Bayesianism) asserts that “quantum mechanics is a tool anyone can use to evaluate, on the basis of one’s past experience, one’s probabilistic expectations for one’s subsequent experience.” (Fuchs, Mermin, and Schack, 2014)

- The inspiration is taken from classical “subjectivist Bayesian” accounts, especially the work of De Finetti

- On such accounts, probabilities are merely the degrees of belief for agents, and the rules of probability arise from the need of such agents to not be incoherent, i.e. to be immune to Dutch books

- A Dutch book for an agent is a collection of bets, each of which the agent will be willing to buy, but which collectively guarantee a sure loss

- It is impossible to construct a Dutch book for an agent if and only if that agent’s credences form a probability measure

- In the quantum context, one can prove analogous results: for example, Randall and Foulis (1976) prove that an agent who assigns probabilities according to the Born rule will avoid “quantum Dutch books”

- The “collapse of the wavefunction” becomes easier to understand in QBism: when a measurement is performed, the quantum state changes to an eigenstate of the measured quantity because the agent updates their expectations about the system

- Similarly, if a measurement is performed on one half of an entangled pair, then the agent updates their expectations about outcomes for the other system; i.e., the state assigned to the other system changes
13.2 The ontological models framework

- QBism contrasts with the other approaches that we have studied, in which the quantum state is to be interpreted as representing some kind of physical fact about the system under investigation.

- Such approaches are often termed $\psi$-ontic.

- Approaches which (like QBism) interpret the quantum state as codifying the expectations of an agent are known as $\psi$-epistemic.

- Following Leifer (2011), we can further distinguish between realist $\psi$-epistemic approaches, in which the wavefunction represents partial information about some underlying ontic state, and anti-realist $\psi$-epistemic approaches, in which the wavefunction represents information but there is no underlying ontic state.

- The ontological models framework of Harrigan and Spekkens (2010) provides a neat way of capturing the distinction between $\psi$-ontic and realist $\psi$-epistemic approaches.

- In this framework, each quantum state $\psi$ is associated with some probability distribution $\mu_\psi$ over ontic states $\lambda \in \Lambda$.

- When we prepare a system in the quantum state $\psi$, it in fact assumes some ontic state $\lambda$ in the support of $\mu_\psi$, with probability $\mu_\psi(\lambda)$.

- The ontic state $\lambda$, in turn, determines a probability distribution over the values of any observable of the system.

- An ontological model is $\psi$-ontic if the probability distributions for distinct quantum states never overlap (i.e. always have disjoint supports): for, in that case, a given ontic state is only ever consistent with a single quantum state.

- An ontological model is $\psi$-epistemic only if there exist distinct quantum states with overlapping probability distributions.

![Figure 13.1: Illustration of the probability distributions in $\psi$-epistemic (left) and $\psi$-ontic (right) models.](image)
13.3 The PBR theorem

- Pusey, Barrett and Rudolph (2012) show that $\psi$-epistemic ontological models will, in general, contradict the predictions of quantum mechanics: that is, that if $\psi$ and $\phi$ are distinct quantum states, and we assume the correctness of the quantum-mechanical predictions, then $\mu_\psi$ cannot overlap with $\mu_\phi$.

- Suppose that we are able to prepare each of a pair of spin-1/2 systems in either the z-spin-up state $\uparrow$ or the x-spin-up state $u = (\uparrow + \downarrow)/\sqrt{2}$.

- By the assumption that $\mu_\uparrow$ and $\mu_u$ overlap, if we prepare the quantum state $\uparrow \otimes \uparrow$, there is some probability that both systems will have an ontic state in the overlap region.

- Now consider the (joint-system) measurement whose eigenstates and eigenvalues are as follows:

\[
\begin{align*}
\frac{1}{\sqrt{2}}(\uparrow \otimes \downarrow + \downarrow \otimes \uparrow) & \quad \text{eigenvalue 1} \\
\frac{1}{\sqrt{2}}(\uparrow \otimes d + \downarrow \otimes u) & \quad \text{eigenvalue 2} \\
\frac{1}{\sqrt{2}}(u \otimes \downarrow + d \otimes \uparrow) & \quad \text{eigenvalue 3} \\
\frac{1}{\sqrt{2}}(u \otimes d + d \otimes u) & \quad \text{eigenvalue 4}
\end{align*}
\]

- Since the eigenvalue-1 outcome is associated with an eigenstate orthogonal to $\uparrow \otimes \uparrow$, the probability of getting this eigenvalue (conditional on preparing that state) must be 0.

- Hence, the ontic states in the overlap region must prescribe a probability of 0 to the eigenvalue 1 (for a measurement of this quantity).

- However, we also have a non-zero probability of obtaining an ontic state in the overlap region if we prepare the quantum state $\uparrow \otimes u$.

- Since this state is orthogonal to the eigenvalue-2 eigenstate, the ontic states in the overlap region must assign probability 0 to getting eigenvalue 2.

- Similarly, we might obtain an ontic state in the overlap region if we prepare $u \otimes \uparrow$ or $u \otimes u$; and hence, the overlap states must assign 0 probability to the eigenvalues 3 and 4.

- But now we have a contradiction: after all, we must surely obtain some outcome when we perform this measurement!

- The PBR proof then generalises this form of argument to arbitrary pairs of states.
A Kochen-Specker dominoes

As a warmup to discussing the Kochen-Specker theorem, the students played a game featuring the “dominoes” shown on the next page. The dominoes should be cut out so that each domino has two digits printed on it: thus, the sheet has three rows on it, with the first row featuring the pairs (1, 2), (1, 5), (1, 3), (1, 7), (2, 5) and (2, 8). The game then has the following objective: to make a pile of dominoes which includes every digit (from 1 to 9) once and only once. The students were divided into groups, and the first group to make such a pile would receive chocolate as a reward.

However, the game is rigged: since there are two digits on each domino, any pile in which each digit features once can only contain an even number of digits; hence, it is impossible to make a pile in which all nine digits feature exactly once. This mirrors the impossibility of non-contextually assigning 1 or 0 to all projectors on a Hilbert space. (Obviously, there’s a bit of an issue about how to prevent students realising this too quickly. The chocolate incentive is partly to encourage people to just start playing the game without spending too much time checking whether it’s possible; even so, my students started to question the rules pretty quickly.)

In more detail: each domino represents one of the observables given in the table in §5.2, with the two digits indicating the two families to which it belongs (labelled with 1 to 9, passing from left to right). Thus, for example, the top-left domino in the sheet below represents the observable (0, 0, 0, 1), which belongs to families 1 and 2. And each digit occurs on four dominoes, reflecting the fact that each family has four members. Trying to make a pile in which each digit features exactly once is therefore analogous to assigning the value 1 to exactly one member of each family; non-contextuality is built in by the fact that a single domino can only be in one pile or the other.

And in case you were wondering—after the game was revealed as rigged, all the students got some chocolate. I’m not a monster.
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B Can Quantum-Mechanical Description of Reality be Considered Complete?

Einstein, Podolsky, and Rosen

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

B.1

Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term complete, the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counterpart in the physical theory. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are
the elements of the physical reality.

The elements of the physical reality cannot be determined by a priori philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.

...it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say \( A \) and \( B \), do not commute, that is, if \( AB \neq BA \), then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this follows that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For if both of them had simultaneous reality—and thus definite values—these values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.

In quantum mechanics it is usually assumed that the wave function does contain a complete description of the physical reality of the system in the state to which it corresponds. At first sight this assumption is entirely reasonable, for the information obtainable from a wave function seems to correspond exactly to what can be measured without altering the state of the system. We shall show, however, that this assumption, together with the criterion of reality given above, leads to a contradiction.

**B.2**

For this purpose let us suppose that we have two systems, I and II, [and] let us designate the corresponding wave function by \( \Psi \). ...

Let \( a_1, a_2, a_3, \cdots \) be the eigenvalues of some physical quantity \( A \) pertaining to system I and \( u_1, u_2, u_3, \cdots \) the corresponding eigenvectors. Then \( \Psi \) can be expressed as

\[
\Psi = \sum_n \psi_n \otimes u_n, \quad (B.7)
\]

where the \( \psi_n \) are vectors used to describe the second system. Suppose now that the
quantity $A$ is measured and it is found that it has the value $a_k$. It is then concluded that after the measurement the first system is left in the state given by the vector $u_k$, and that the second system is left in the state given by the vector $\psi_k$. This is the process of reduction of the wave packet; the vector given by the infinite series (7) is reduced to a single term $u_k \otimes \psi_k$.

The set of vectors $u_n$ is determined by the choice of the physical quantity $A$. If, instead of this, we had chosen another quantity, say $B$, having the eigenvalues $b_1, b_2, b_3, \cdots$ and eigenvectors $v_1, v_2, v_3, \cdots$ we should have obtained, instead of Eq. (7), the expansion

$$\Psi = \sum_n \phi_n \otimes v_n \quad \text{(B.8)}$$

If now the quantity $B$ is measured and is found to have the value $b_r$, we conclude that after the measurement the first system is left in the state given by $v_r$ and the second system is left in the state given by $\phi_r$.

We see therefore that, as a consequence of two different measurements performed upon the first system, the second system may be left in states with two different state vectors. On the other hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system. This is, of course, merely a statement of what is meant by the absence of an interaction between the two systems. Thus, it is possible to assign two different state vectors (in our example $\psi_k$ and $\phi_r$) to the same reality (the second system after the interaction with the first).

Now, it may happen that the two state vectors, $\psi_k$ and $\phi_r$, are eigenvectors of two non-commuting operators corresponding to some physical quantities $P$ and $Q$, respectively. That this may actually be the case can best be shown by an example. Let us suppose that the two systems are two particles, and that in the z-spin basis,

$$\Psi = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \quad \text{(B.9)}$$

[If $A$ is the z-spin of the first particle, then the corresponding $\psi_k$ must be an eigenvector of the z-spin of the second particle. If $B$ is the x-spin of the first particle, then the corresponding $\phi_r$ must be an eigenvector of the x-spin of the second particle. Hence,] we have shown that it is in general possible for $\psi_k$ and $\phi_r$ to be eigenvectors of two noncommuting operators, corresponding to physical quantities.

Returning now to the general case contemplated in Eqs. (7) and (8), we assume that $\psi_k$ and $\phi_r$ are indeed eigenvectors of some noncommuting operators $P$ and $Q$, corresponding to the eigenvalues $p_k$ and $q_r$, respectively. Thus, by measuring either $A$ or $B$ we are in a position to predict with certainty, and without in any way disturbing the second system, either the value of the quantity $P$ (that is $p_k$) or the value of the quantity $Q$ (that is $q_r$). In accordance with our criterion of reality, in the first case we must consider the quantity $P$ as being an element of reality, in the second case the
quantity $Q$ is an element of reality. But, as we have seen, both wave functions $\psi_k$ and $\phi_r$ belong to the same reality.

Previously we proved that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (1) leads to the negation of the only other alternative (2). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.

One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted. On this point of view, since either one or the other, but not both simultaneously, of the quantities $P$ and $Q$ can be predicted, they are not simultaneously real. This makes the reality of $P$ and $Q$ depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this.

While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

Questions

1. “According to EPR, if every element of physical reality has a counterpart in the theory, then the theory is complete.” True or false?
2. What is EPR’s criterion of reality?
3. Make an argument map of the argument in paragraph (α).
4. Make an argument map of the argument in paragraph (β).
5. Express the state $\Psi$ (Eq. 9) in terms of x-spin eigenvectors. Use this to satisfy yourself that if the x-spin of the first particle is measured, the post-measurement state of the second particle must be an eigenvector of x-spin.
6. Reconstruct the logical structure of the paragraph labelled (γ).
7. Make an argument map of the paper as a whole.
C Computing correlations

Suppose that Alice and Bob prepare a system in the EPR state:

\[ \Psi = \frac{1}{\sqrt{2}} (\uparrow_A \otimes \downarrow_B - \downarrow_A \otimes \uparrow_B) \]  

(C.1)

(The A and B subscripts aren’t really necessary, but they’re there to help us keep track of which system is Alice’s and which is Bob’s.) Here, \( \uparrow \) and \( \downarrow \) are the eigenstates for the z-spin operator \( Z \), with eigenvalues +1 and −1 respectively:

\[ Z(\uparrow) = \uparrow \]  
\[ Z(\downarrow) = -\downarrow \]  

(C.2)  
(C.3)

It follows that \( \uparrow_A \otimes \downarrow_B \) and \( \downarrow_A \otimes \uparrow_B \) are both eigenstates of the joint operators \( Z_A \otimes I_B \) and \( I_A \otimes Z_B \) (where \( I \) is the identity operator):

\[ Z_A \otimes I_B(\uparrow_A \otimes \downarrow_B) = \uparrow_A \otimes \downarrow_B \]  
\[ I_A \otimes Z_B(\uparrow_A \otimes \downarrow_B) = -\uparrow_A \otimes \downarrow_B \]  

(C.4)

\[ Z_A \otimes I_B(\downarrow_A \otimes \uparrow_B) = -\downarrow_A \otimes \uparrow_B \]  
\[ I_A \otimes Z_B(\downarrow_A \otimes \uparrow_B) = \downarrow_A \otimes \uparrow_B \]  

(C.5)

Thus, (C.4) indicates that \( \uparrow_A \otimes \downarrow_B \) is a state in which Alice will (definitely) get a result of +1 on measuring z-spin, whilst Bob will (definitely) get a result of −1 on measuring z-spin; similarly, (C.5) shows that Alice and Bob will get −1 and +1, respectively, on measuring z-spin. Hence, the state \( \Psi \) in (C.1) is associated with a 50% chance that Alice measures +1 and Bob measures −1, and a a 50% chance of the other way round.

Now we turn to the probabilities arising from other measurements that Alice and Bob could make. We will denote the eigenstates of the x-spin operator \( X \) by \( u \) and \( d \), so that

\[ X(u) = u \]  
\[ X(d) = -d \]  

(C.6)  
(C.7)

We will make use of the fact (not proved here) that

\[ \uparrow = \frac{1}{\sqrt{2}}(u + d) \]  
\[ \downarrow = \frac{1}{\sqrt{2}}(u - d) \]  

(C.8)  
(C.9)

---

1Strictly speaking, this isn’t “the” EPR state, since the EPR paper doesn’t use spins; but it’s analogous to the state that EPR use.
First, if we express $\Psi$ in terms of $z$-spin eigenstates for Alice and $x$-spin eigenstates for Bob, we find:

$$\Psi = \frac{1}{\sqrt{2}} \left( \uparrow_A \otimes \frac{1}{\sqrt{2}}(u_B - d_B) - \downarrow_A \otimes \frac{1}{\sqrt{2}}(u_B + d_B) \right) \quad \text{(C.10)}$$

$$= \frac{1}{2} \left( (\uparrow_A \otimes u_B) - (\uparrow_A \otimes d_B) - (\downarrow_A \otimes u_B) - (\downarrow_A \otimes d_B) \right) \quad \text{(C.11)}$$

By the same reasoning as above, the four terms in the sum (C.11) are associated—respectively—with Alice and Bob obtaining the outcomes $(+1, +1)$, $(+1, -1)$, $(-1, +1)$ and $(-1, -1)$ when Alice measures $z$-spin and Bob measures $x$-spin. Hence, there is a $1/4$ probability of obtaining each of these outcomes. If we express $\Psi$ in terms of $x$-spin eigenstates for Alice and $z$-spin eigenstates for Bob, then by an analogous computation we find that there is also a $1/4$ probability of obtaining each possible outcome.

Finally, if $\Psi$ is expressed in terms of $x$-spin eigenstates for both Alice and Bob then we find that

$$\Psi = \frac{1}{2\sqrt{2}} \left( u_A \otimes u_B + d_A \otimes u_B - u_A \otimes d_B - d_A \otimes d_B \right. \right.$$

$$\left. - u_A \otimes u_B + d_A \otimes u_B - u_A \otimes d_B + d_A \otimes d_B \right) \quad \text{(C.12)}$$

$$= \frac{1}{\sqrt{2}} (d_A \otimes u_B - u_A \otimes d_B) \quad \text{(C.13)}$$

Hence, by the same reasoning as we used for $z$-spin, we conclude that there is a $50\%$ chance of Alice measuring $+1$ and Bob measuring $-1$, a $50\%$ chance of the converse, and no chance of them measuring the same spin.

Hence, the correlation table for the EPR measurements is as follows:

<table>
<thead>
<tr>
<th>Bob</th>
<th>Alice</th>
<th>$z$-spin ($Z_A$)</th>
<th>$x$-spin ($X_A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$-spin ($Z_B$)</td>
<td>$+1$</td>
<td>$0$</td>
<td>$1/2$</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$1/2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x$-spin ($X_B$)</td>
<td>$+1$</td>
<td>$1/4$</td>
<td>$1/4$</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$1/4$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

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