10 Bohmian mechanics

Wavefunctions

- The state of a single (and spinless) particle is given by a wavefunction: a map

\[ \psi : T \times \Sigma \rightarrow \mathbb{C} \]  

where \( T \) and \( \Sigma \) represent time and space

- Given a particle with wavefunction \( \psi \), if a measurement is performed at time \( t \) to determine if the particle is in some region \( \Delta \subseteq \Sigma \), then the probability of getting a positive answer is

\[ \int_{x \in \Delta} |\psi(t, x)|^2 \, dx \]  

(10.2)

- In other words, the wavefunction defines a probability density

\[ \rho(t, x) = |\psi(t, x)|^2 \]  

(10.3)

- Like any probability density, this is required to be normalised: for any time \( t \),

\[ \int_{x \in \Sigma} |\psi(t, x)|^2 \, dx = 1 \]  

(10.4)

- On standard approaches to QM, if we measure a particle as being in the region \( \Delta \), then immediately after the measurement its wavefunction is given by the (normalised) restriction of \( \psi \) to \( \Delta \)

- Outside measurement contexts, the wavefunction evolves according to Schrödinger's equation:

\[ \frac{d\psi}{dt} = H\psi \]  

(10.5)

where \( H \) is the Hamiltonian

- For example, the Hamiltonian for a particle of mass \( m \) in a potential described by a function \( V : \Sigma \rightarrow \mathbb{R} \) is

\[ H = \frac{1}{2m} \nabla^2 + V \]  

(10.6)

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \)
The guidance equation

- The fundamental entities of Bohmian mechanics are the particles: pointlike objects, which have definite positions at all times, and which are held to be the fundamental constituents of macroscopic matter
- Given a single particle of mass \( m \), its position \( Q \in \Sigma \) evolves deterministically over time in a manner determined by the wavefunction \( \psi : \Sigma \to \mathbb{C} \), as described by the guidance equation:
  \[
  \frac{dQ}{dt} = \frac{\hbar}{m} \text{Im} \left( \frac{\nabla \psi}{\psi} (Q) \right)
  \]  
  (10.7)
- Since the guidance equation is first-order, the wavefunction determines the particle’s velocity
- The evolution of the particle depends on the evolution of the wavefunction, but not vice versa: the wavefunction always evolves according to the usual Schrödinger equation (10.5), even in measurement contexts
- Note that the probability density \( |\psi(x)|^2 \) is equivariant: evolving the wavefunction via (10.5) delivers the same result as evolving the probability density via (10.7)
- In each run of the two-slit experiment the particle goes through one slit or another, but the interference within the wavefunction “guides” the particle towards certain parts of the screen and away from others (see Figure 10.1)

![Figure 10.1: The Bohmian trajectories for the two-slit experiment.](image)

- The outcome of a non-position measurement depends, in general, on the method of measurement used: this provides a sense in which the theory is contextual
The notion of primitive ontology

Bell, *The Theory of Local Beables* (1975):

The concept of ‘observable’ lends itself to very precise mathematics when identified with ‘self-adjoint operator’. But physically, it is a rather woolly concept. [...] So it could be hoped that some increase in precision might be possible by concentration on the beables, which can be described in classical terms, because they are there.


[...] in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the ‘measurement’ of anything else, then you commit redundancy and risk inconsistency.


According to (pre-quantum-mechanical) scientific precedent, when new mathematically abstract theoretical entities are introduced into a theory, the physical significance of these entities, their very meaning insofar as physics is concerned, arises from their dynamical role, from the role they play in (governing) the evolution of the more primitive—more familiar and less abstract—entities or dynamical variables. [...] Why would these abstractions be introduced in the first place, if not for their relevance to the behavior of something else, which somehow already has physical significance?

Several particles

- If we have $N$ particles, then their collective state at any given time may be represented by an $N$-tuple $(Q_1, \ldots, Q_N) \in \Sigma^N$, i.e. by a point in configuration space

- This time, the guidance equation describes the evolution of the collection of particles, as determined by the wavefunction $\Psi : \Sigma^N \to \mathbb{C}$

$$\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \text{Im} \left( \nabla_i \frac{\Psi}{\bar{\Psi}} (Q_1, \ldots, Q_N) \right)$$

(10.8)

where $m_i$ is the mass of the $i$th particle, and $\nabla_i$ denotes the position-derivative associated with the $i$th factor of $\Sigma^N$.

- Note that this dynamics is non-local, since the wavefunction is a function on configuration space (and the motion of any one particle depends on the wavefunction's value at the “occupied” point of configuration space)
Subsystems

- Suppose that we have some system of $N$ particles governed by Bohmian mechanics, as described above.
- Now say the first $M < N$ particles are a subsystem, with the remaining particles being the environment.
- Let $x$ be coordinates for $\Sigma^M$ and $y$ be coordinates for $\Sigma^{N-M}$, and write $Q = (X, Y)$.
- Given a particular environment configuration $Y \in \Sigma^{N-M}$, define the conditional wavefunction of the subsystem as
  
  $$\psi(x) := \Psi(x,Y)$$

  \hfill (10.9)

- The conditional wavefunction determines the conditional probability distribution for the subsystem configuration (conditional, that is, on the environment configuration):
  
  $$\overline{\rho}(x|Y) = |\psi(x)|^2$$

  \hfill (10.10)

- Note that the conditional wavefunctions of subsystem and environment underdetermine the full system wavefunction $\Psi$.
- Now suppose that the total-system wavefunction is of the form
  
  $$\Psi(x, y) = \chi(x)\phi(y) + \Psi_\perp(x, y)$$

  \hfill (10.11)

  where $\Psi_\perp$ and $\phi$ have disjoint support in the environment coordinates $y$.
- If $Y \in \text{supp}(\phi)$, then the guidance equation for $X$ reduces to
  
  $$\frac{dX_i}{dt} = \frac{\hbar}{m_i} \text{Im} \left( \nabla_i \chi(X) \right)$$

  \hfill (10.12)

  and we refer to $\chi$ as the effective wavefunction for the subsystem.
- Moreover, observe that if $Y \in \text{supp}(\phi)$, then $\chi = \psi$ (i.e., the effective wavefunction coincides with the conditional wavefunction).
- Replacing the total-system wavefunction by its conditional/effective wavefunction is referred to as a process of “effective collapse”; this is what justifies the (informal) use of the collapse postulate by Bohmians.
- It is in this sense that one can make use of Bohmian mechanics, even in the absence of data about the universal wavefunction.
- It also means that effective wavefunctions may be very different from the universal wavefunction; one could have time-dependent effective wavefunctions yet a time-independent universal wavefunction, for example.