Hilbert space: problems
Quantum Foundations, WS2019/20

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Exercise 1
Show that an operator $O$ on $\mathbb{C}^n$ (equipped with the standard inner product) is symmetric just in case $O_{ij} = O_{ji}$.

Exercise 2
1. Show that the identity map from $\mathbb{C}^2$ to itself is given by the matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

(1)

2. More generally, show that the identity map from $\mathbb{C}^n \to \mathbb{C}^n$ is expressed by the matrix

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

(2)

Exercise 3
1. Let $F$ be a linear map from $\mathbb{C}^n$ to $\mathbb{C}^p$ (expressed by an $p \times n$ matrix $F_{ji}$), and let $G$ be a linear map from $\mathbb{C}^p$ to $\mathbb{C}^r$ (expressed by a $r \times p$ matrix $G_{kj}$). Show that the linear map $GF : \mathbb{C}^n \to \mathbb{C}^r$ is expressed by the matrix

\[
(GF)_{ki} = \sum_{j=1}^{p} G_{kj} F_{ji}
\]

(3)
2. Let \( n = 3 \), \( p = 2 \), and \( r = 4 \), and suppose that \( F \) and \( G \) are given by the matrices

\[
(F_{ji}) = \begin{pmatrix}
1 & 0 \\
-3 & 5 \\
4 & -1 \\
7 & 0
\end{pmatrix}, \quad (G_{kj}) = \begin{pmatrix}
0 & 2 & 3 \\
5 & -6 & 1
\end{pmatrix}
\]

Compute the matrix for \( GF \).

**Exercise 4**

In the complex vector space \( \mathbb{C}^2 \), consider the \( x \)-spin and \( y \)-spin operators

\[
S_x = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad S_y = \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}
\]

What are the eigenvalues of \( S_x \) and \( S_y \)? What are their associated eigenvectors (up to phase)?

**Exercise 5**

Prove that the eigenvalues of a symmetric operator are all real.

**Exercise 6**

Suppose that an electron is in an eigenstate of \( z \)-spin-up (i.e., the state \( (1, 0) \)). What is the probability of a spin-up (+1) result if its \( x \)-spin is measured?

**Exercise 7**

1. Show that for any subspace \( J \subseteq H \), \( P_J^2 = P_J \) (where \( P_J^2 := P_J P_J \)).
2. Given any \( \psi \) in some Hilbert space \( H \), show that the projector \( P_\psi \) is given by

\[
P_\psi(\phi) = \frac{\langle \psi, \phi \rangle}{|\psi|^2} \psi
\]

Use this formula to show that \( P_\psi \) is symmetric.

**Exercise 8**

Let \( \varepsilon_q \) be an eigenstate of a non-degenerate eigenvalue \( q \). Show that

\[
\langle \psi, P_q \psi \rangle = |\langle \psi, \varepsilon_q \rangle|^2
\]