5 The Kochen-Specker Theorem

5.1 Von Neumann’s no-hidden-variables theorem

- If $E(Q)$ is the expectation value of a physical quantity $Q$ in some state, then we say that that state is dispersion-free with respect to $Q$ if $E(Q^2) = E(Q)^2$.

- In classical probability theory, states are probability densities and pure states are characteristic functions.

- Hence, classical pure states are dispersion-free with respect to all quantities: they assign a definite value to each such quantity.

- For a quantum system represented by a finite-dimensional Hilbert space $H$, physical quantities are represented by symmetric operators on $H$.

- The usual states we consider (vectors and density operators) have dispersion with respect to at least some quantities.

- Could there be “complete” states which assign definite values to (are dispersion-free with respect to) all quantities?

- An influential negative argument was given by von Neumann in 1935.

- First, assume that any state (complete or otherwise) must assign expectation values to quantities in a linear fashion: for any symmetric operators $Q_1$ and $Q_2$, and any $\alpha, \beta \in \mathbb{R}$:

$$E(\alpha Q_1 + \beta Q_2) = \alpha E(Q_1) + \beta E(Q_2)$$ (5.1)

- Second, assume that the definite value any hidden state assigns to a quantity must be an eigenvalue of the corresponding operator.

- Then there can be no such hidden states, since (in general) the eigenvalues of a sum of operators are not the sums of the eigenvalues of those operators.

- For example, if

$$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then $\frac{1}{\sqrt{2}}(S_x + S_z)$ has eigenvalues $\pm 1$, but $\frac{1}{\sqrt{2}}(\pm 1 \pm 1)$ is either $\sqrt{2}, 0,$ or $-\sqrt{2}$.

- However, Bell (and others) have criticised this proof for its requirement that linearity hold even for incompatible (non-commuting) operators, since such operators cannot be simultaneously measured.
5.2 The Kochen-Specker theorem

- For the Kochen-Specker theorem, we assume that a hidden-variables state would assign definite (eigen)values to operators in such a way that for compatible (commuting) operators, (5.1) holds.

- It can then be shown that if $H$ has at least three dimensions, no such hidden-variables model is possible.

- For this proof of the theorem, we suppose that $H$ is four-dimensional (which will prove the theorem for four or more dimensions).

- Suppose that $P_1, P_2, P_3$ and $P_4$ are projectors onto four orthogonal one-dimensional subspaces of $H$.

- Then the $P_i$ are all compatible, and $I = \sum_i P_i$.

- It follows that the hidden-variables state must assign 1 to exactly one of the $P_i$, whilst the other three receive the value 0.

- We then consider the projectors onto the following 18 subspaces, organised into 9 overlapping families of four orthogonal subspaces each:

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & 0 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

- There is no satisfactory way to assign 1s and 0s, provided that we assign the same value to the same projector in two different families (non-contextuality).