Algebras of quantities: problems
Quantum Foundations, WS2019/20

Neil Dewar

Exercise 1

Let $\rho$ be a probability density over a finite set $W$, and consider the *-algebra $\mathcal{C}(W)$ of functions $f : W \to \mathbb{C}$.

1. Show that the map $\mathcal{E} : \mathcal{C}(W) \to \mathbb{C}$ given by

$$\mathcal{E}(f) = \sum_{w \in W} \rho(w) f(w) \quad (1)$$

is a state on this algebra (i.e. that it is positive, linear, and unit-preserving).

2. For any subset $U \subseteq W$, let $\chi_U : W \to \mathbb{C}$ be the characteristic function of $U$: that is, the function given by

$$\chi_U(w) = \begin{cases} 1 & \text{if } w \in U \\ 0 & \text{if } w \not\in U \end{cases} \quad (2)$$

Show that

$$\mathcal{E}(\chi_U) = \rho(U) \quad (3)$$

where $\rho(U)$ is the probability measure of $U$ (i.e., $\rho(U) = \sum_{w \in U} \rho(w)$).

Exercise 2

Let $\mathcal{C}(W)$ be the *-algebra of functions $f : W \to \mathbb{C}$ for some finite set $W$. For any $v \in W$, let $\chi_v := \chi_{\{v\}}$.

1. Show that for any state $\omega$ on $\mathcal{C}(W)$, the function $\rho : W \to [0, 1]$ given by

$$\rho(v) = \omega(\chi_v) \quad (4)$$

is a probability density. You may assume that if $f$ is real-valued, then $\omega(f) \in \mathbb{R}$.
2. Show that for any \( f \in \mathcal{C}(W) \), and for \( \rho \) defined by equation (4), \[
\omega(f) = \sum_{v \in W} \rho(v)f(v)
\] (5)

(Hint: try expressing \( f \) in terms of the characteristic functions \( \chi_v \).)

**Exercise 3**

Let \( \psi \) be a unit vector in some Hilbert space \( H \), and consider the *-algebra \( \mathcal{B}(H) \) of operators on \( H \). Show that the map \( \mathcal{E}_\psi : \mathcal{B}(H) \to \mathbb{C} \) given by \[
\mathcal{E}_\psi(Q) = \langle \psi, Q\psi \rangle
\] (6)
is a state on this algebra (i.e. that it is positive, linear, and unit-preserving).

**Exercise 4**

1. Let \( A \) be an arbitrary operator on \( \mathbb{C}^n \), hence expressible as a matrix \( A_{ij} \). Prove that the adjoint \( A^* \) of \( A \) is given by the “conjugate transpose” matrix, \( A_{ij}^* \).

2. Use this result to prove that axiom (4.8) holds for operators on \( \mathbb{C}^n \). (You should also reflect upon the other axioms, to convince yourself that they also hold for such operators.)

**Exercise 5**

Show that the following operators on \( \mathbb{C}^2 \) do not commute with one another:

\[
S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

**Exercise 6**

Given a state \( \omega \) on a *-algebra \( \mathfrak{A} \), the dispersion of any \( X \in \mathfrak{A} \) is defined as \( \omega(X^2) - \omega(X)^2 \).

1. Let \( W \) be a finite set, and let \( \chi_v \) be as in Exercise 2 (note that \( \chi_v \) is a probability density over \( W \)). Show that for any \( f : W \to \mathbb{C} \), the dispersion of \( f \) relative to \( \chi_v \) is 0.

2. Let \( \uparrow \) be shorthand for the vector \( (1, 0) \in \mathbb{C}^2 \), and consider the state \( \mathcal{E}_\uparrow \) on \( \mathcal{B}(\mathbb{C}^2) \) (defined as in equation (6)). Compute the dispersions of \( S_x \) and \( S_x^2 \).