4 Algebras of quantities

4.1 *-algebras

- A *-algebra $\mathcal{A}$ is a complex vector space, equipped with the following further structure:
  - A binary operation of multiplication: we denote the product of $X, Y \in \mathcal{A}$ by $XY$
  - A special element $1 \in \mathcal{A}$
  - A unary operation of involution: we denote the involute of $X \in \mathcal{A}$ by $X^*$

These operations are required to obey the following axioms (in addition to the vector space axioms): for any $X, Y, Z \in \mathcal{A}$ and $\alpha \in \mathbb{C}$,

\begin{align}
  (XY)Z &= X(YZ) \\
  X1 &= X = 1X \\
  X(Y + Z) &= XY + XZ \\
  (X + Y)Z &= XZ + YZ \\
  X^{**} &= X \\
  (X + Y)^* &= (X^* + Y^*) \\
  (\alpha X)^* &= \overline{\alpha}X^* \\
  (XY)^* &= Y^*X^* \\
  1^* &= 1
\end{align}

- A C*-algebra is a *-algebra obeying some further conditions, which we will not discuss: instead, we will confine our attention to finite-dimensional *-algebras, which are guaranteed to be C*-algebras

- A state on $\mathcal{A}$ is a positive, unit-preserving, linear functional over $\mathcal{A}$: that is, a linear map $\omega: \mathcal{A} \to \mathbb{C}$ such that for all $X \in \mathcal{A}$,

\begin{align}
  \omega(X^*X) &\geq 0 \quad \text{(hence } \omega(X^*X) \in \mathbb{R}) \\
  \omega(1) &= 1
\end{align}

- An element $X \in \mathcal{A}$ is said to be self-adjoint if $X^* = X$

- If $X$ is self-adjoint, then $\omega(X) \in \mathbb{R}$ for any state $\omega$
4.2 Commuting *-algebras

- Let $W$ be a finite set; then the set $\mathcal{C}(W)$ of all functions $f : W \to \mathbb{C}$ is a *-algebra, where 1 is the constant function $w \mapsto 1$ and the operations are defined as follows: for any $w \in W$,
  \[
  (fg)(w) = f(w) \cdot g(w) \quad (4.12) \\
  f^*(w) = f(w) \quad (4.13)
  \]

- The self-adjoint elements of this *-algebra are the real-valued functions on $W$
- For any probability density $\rho$ on $W$, the expectation-value map $f \mapsto \sum_{w \in W} f(w)p(w)$ is a state
- Moreover, any state $\mathcal{C} \to \mathbb{R}$ is the expectation-value map for some probability density: so states can be identified with probability densities
- Note that $\mathcal{C}$ is commutative: for any $f, g \in \mathcal{C}$, $fg = gf$
- Any finite-dimensional commuting *-algebra can be represented as an algebra of complex functions

4.3 Quantum *-algebras

- Let $H$ be a finite-dimensional Hilbert space; then the set $\mathfrak{B}(H)$ of all operators on $H$ is a *-algebra, where 1 is the identity operator and multiplication is defined as composition
- Involution is defined via the notion of adjoint: for any operator $O$, its adjoint is the (unique) operator $O^*$ such that for any $\phi, \psi \in H$,
  \[
  \langle \phi, O^* \psi \rangle = \langle O\phi, \psi \rangle \quad (4.14)
  \]

- The self-adjoint elements of this *-algebra are the symmetric operators
- For any density operator $\rho$ over $H$, the expectation-value map $Q \mapsto \text{Tr}(\rho Q)$ is a state
- Moreover, if $\dim(H) \geq 3$, then any state $\mathfrak{B}(H) \to \mathbb{R}$ is the expectation-value map for some density operator: so states can be identified with density operators
- Note that such a *-algebra will not, in general, be commutative
- Any finite-dimensional *-algebra can be represented as an algebra of operators (i.e., as a subalgebra\(^1\) of $\mathfrak{B}(H)$, for some Hilbert space $H$)

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\(^1\)A subalgebra of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ which is also a *-algebra: that is, a subspace which contains 1, and is closed under multiplication and involution.