The problems below are taken or adapted from:


**Exercise 1**

Let $V$ be a (real or complex) vector space. Show that for any $u, v \in V$ and $\alpha \in \mathbb{C}$:

1. If $u + u = u$, then $u = 0$.
2. If $u + v = 0$, then $u = -v$.
3. $(-1)u = -u$
4. $\alpha 0 = 0$

**Answer of exercise 1**

1. 

\[
\begin{align*}
    u + u &= u \\
    (u + u) + (-u) &= u + (-u) \\
    u + (u + (-u)) &= u + (-u) \\
    u + 0 &= 0 \\
    u &= 0
\end{align*}
\]
2.

\[ u + v = 0 \]
\[ (u + v) + (-v) = 0 + (-v) \]
\[ u + (v + (-v)) = -v \]
\[ u + 0 = -v \]
\[ u = -v \]

3.

\[ u + (-1)u = 1u + (-1)u \]
\[ = (1 - 1)u \]
\[ = 0u \]
\[ = 0 \]

Hence, by the previous question, \((-1)u = -u\).

4.

\[ \alpha 0 = \alpha (u + (-u)) \]
\[ = \alpha ((1 - 1)u) \]
\[ = \alpha (0u) \]
\[ = (\alpha 0)u \]
\[ = 0u \]
\[ = 0 \]

Exercise 2

Let \( V \) be a vector space, and let \( X \) be an arbitrary set. Show that the set of functions \( \{f : X \rightarrow V\} \) is a vector space, with addition and scalar multiplication defined by

\[ (f + g)(x) = f(x) + g(x) \]  \hspace{1cm} (1)
\[ (\alpha f)(x) = \alpha (f(x)) \]  \hspace{1cm} (2)
Answer of exercise 2

First, we need to also define the zero element and the inversion operation. We define the zero as the function \( z \) such that for all \( x \in X \), \( z(x) = 0 \) (0 being the zero element in \( V \)). We define the inverse of any \( f : X \to V \) as the function \(-f\) such that \((-f)(x) = -(f(x))\).

From here, it is just a question of showing that the axioms hold, by using the fact that they hold in \( V \). For example, the proof of (2.8) goes as follows: for any \( f, g : X \to V \), \( \alpha \in \mathbb{C} \), and \( x \in X \),

\[
\alpha(f + g)(x) = \alpha((f + g)(x))
\]

\[
= \alpha(f(x) + g(x))
\]

\[
= \alpha f(x) + \alpha g(x)
\]

\[
= (\alpha f + \alpha g)(x)
\]

Exercise 3

Show that any subspace \( U \) of \( V \) must be closed under inversion, and must contain the zero element.

Answer of exercise 3

First, \( U \) is closed under scalar multiplication, for any element \( u \in U \), \((-1)u = -u\) is also in \( U \). Second, since \( U \) is closed under addition (and under inversion), and contains at least one element \( u \), it must contain \( 0 = u + (-u) \).

Exercise 4

Let \( U \) and \( V \) be subspaces of the vector space \( W \). Show that \( U \cap V \) (the intersection of \( U \) and \( V \)) is a subspace of \( W \).

Answer of exercise 4

By the above, \( 0 \) is in both \( U \) and \( V \), and hence in \( U \cap V \). So we must show that \( U \cap V \) is closed under addition and scalar multiplication.

For addition: if \( u, v \in U \cap V \), then \( u, v \in U \) and \( u, v \in V \); hence \( u+v \in U \) and \( u+v \in V \) (since \( U \) and \( V \) are both closed under addition); hence \( u+v \in U \cap V \).

For scalar multiplication: if \( u \in U \cap V \), then \( u \in U \) and \( u \in V \); hence \( \alpha u \in U \) and \( \alpha u \in V \); hence \( \alpha u \in U \cap V \).
Exercise 5

Let $H$ be a Hilbert space (i.e., complex vector space with inner product).

1. Show that for any $x, y, z \in H$,

$$\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$$  \hspace{1cm} (3)

2. Show that for any $x, y \in H$,

$$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$$  \hspace{1cm} (4)

(recalling that $|x|^2 = \langle x, x \rangle$).

Answer of exercise 5

1. 

$$\langle \alpha x + \beta y, z \rangle = \overline{\langle z, \alpha x + \beta y \rangle} = \overline{\alpha \langle z, x \rangle + \beta \langle z, y \rangle} = \overline{\alpha} \overline{\langle z, x \rangle} + \overline{\beta} \overline{\langle z, y \rangle} = \overline{\alpha \langle x, z \rangle} + \overline{\beta \langle y, z \rangle}$$

2. 

$$|x + y|^2 + |x - y|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle = 2(|x|^2 + |y|^2)$$
Exercise 6

Consider the complex vector space $\mathbb{C}^n$. Show that an inner product may be defined by the following formula: for any $\vec{x}, \vec{y} \in \mathbb{C}^n$,

$$\langle \vec{x}, \vec{y} \rangle := \sum_i x_i y_i$$  \hspace{1cm} (5)

(where $\vec{x} = (x_1, \ldots, x_n)$. We will call this the standard inner product on $\mathbb{C}^n$.

Answer of exercise 6

Axiom (2.11): for any $\vec{x}, \vec{y} \in \mathbb{C}^n$,

$$\langle \vec{y}, \vec{x} \rangle = \sum_i y_i x_i = \sum_i \overline{y_i} x_i = \sum_i x_i y_i = \langle \vec{x}, \vec{y} \rangle$$

Axiom (2.12): for any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$,

$$\langle \vec{x}, \alpha \vec{y} + \beta \vec{z} \rangle = \sum_i x_i (\alpha y_i + \beta z_i)$$

$$= \alpha \sum_i x_i y_i + \beta \sum_i x_i z_i = \alpha \langle \vec{x}, \vec{y} \rangle + \beta \langle \vec{x}, \vec{z} \rangle$$

Axiom (2.13): for any $\vec{x} \in \mathbb{C}$,

$$\langle \vec{x}, \vec{x} \rangle = \sum_i x_i x_i = \sum_i |x_i|^2$$

If $\vec{x} \neq 0$, then for some $i$, $x_i \neq 0$; hence $|x_i|^2 > 0$, and so $\langle \vec{x}, \vec{x} \rangle > 0$. 


Exercise 7

Consider the complex vector space \( \mathbb{C}^2 \), and equip it with the standard inner product. Show that the following pairs of vectors form orthonormal bases for \( \mathbb{C}^2 \):

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}
\]

Answer of exercise 7

This is mostly just a matter of computation. Note that a little care needs to be taken with the third pair, since we have to keep track of complex conjugation: for instance, the inner product of the two vectors is

\[
\left( \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) + \left( \frac{i}{\sqrt{2}} \right) \left( -\frac{i}{\sqrt{2}} \right) = \frac{1}{2} + \frac{1}{2} \text{i}^2
\]

\[
= 0
\]