0 A primer on complex numbers

0.1 The complex numbers

- The complex numbers \( \mathbb{C} \) are an algebraic extension of the real numbers \( \mathbb{R} \), used to let us take square roots of negative numbers

- We define them by first introducing the “imaginary” number \( i \), stipulated to be such that \( i^2 = -1 \), and letting complex numbers be linear combinations ("complexes") of \( i \) and 1

- That is, a complex number is anything of the form \( a + bi \), where \( a, b \in \mathbb{R} \)

- We define addition and multiplication of complex numbers in terms of addition and multiplication of real numbers:

\[
(a + bi) + (c + di) := (a + c) + (b + d)i \\
(a + bi)(c + di) := (ac - bd) + (bc + ad)i
\]  

(0.1) (0.2)

- Note that with these definitions, addition and multiplication are commutative and associative, and multiplication distributes over addition

- We also introduce the operation of complex conjugation: the complex-conjugate of \( z \in \mathbb{C} \) is denoted \( \overline{z} \), and conjugation is defined by

\[
\overline{a + bi} := a - bi
\]  

(0.3)

- One can show that the following identities hold for any \( x, y \in \mathbb{C} \):

\[
\overline{x} = x \\
\overline{x + y} = \overline{x} + \overline{y} \\
\overline{xy} = \overline{x}(\overline{y})
\]  

(0.4) (0.5) (0.6)

- One can also show that \( \overline{x} = x \) iff \( x \in \mathbb{R} \), and \( \overline{x} = -x \) iff \( x = bi \) for some \( b \in \mathbb{R} \)

- We define the modulus of a complex number by

\[
|z| := \sqrt{zz}
\]  

(0.7)

- Note that if \( z = a + bi \) then \( \overline{z}z = a^2 + b^2 \), so \( \overline{z}z \) (and hence \(|z|\)) is always a positive real number
• For any complex number $z = a + bi$, if we define the phase $\phi := \tan^{-1}\left(\frac{b}{a}\right)$, then

$$z = |z| \cos \phi + (|z| \sin \phi)i$$ (0.8)

• This is referred to as the polar representation of $z$

• Defining $r := |z|$ and $e^{i\phi} := \cos \phi + i \sin \phi$, we often write this as $z = re^{i\phi}$

• Note that multiplying complex numbers is easier in the polar representation:

$$(re^{i\phi})(se^{i\psi}) = (rs)e^{i(\phi + \psi)}$$ (0.9)

0.2 The complex plane

• It is often convenient to represent the complex numbers on a plane, with the real numbers as the horizontal axis and the imaginary numbers as the vertical axis:

• Complex conjugation corresponds to reflection in the real axis

• The modulus of a complex number $z = re^{i\phi}$ corresponds to its distance from the origin, and the phase to the angle it makes with the real axis: