Metric vector spaces

Formally, a metric vector space \( V \) comprises the following data:

- a set \( V \) \((\text{of vectors})\)
- a privileged element \( 0 \) of \( V \) \((\text{the zero vector})\)
- a binary operation \( + : V \times V \to V \) \((\text{addition})\)
- a binary operation \( \cdot : \mathbb{R} \times V \to V \) \((\text{scalar multiplication})\)
- a binary operation \( \langle , \rangle : V \times V \to \mathbb{R} \) \((\text{an inner product})\)

The structure \( V = (V, 0, +, \cdot, \langle , \rangle) \) must obey the following axioms:

(V1) For all \( u, v \in V \), \( u + v = v + u \)
(V2) For all \( u, v, w \in V \), \( (u + v) + w = u + (v + w) \)
(V3) For all \( u \in V \), \( u + 0 = u \)
(V4) For all \( u \in V \), there is a \( v \in V \) such that \( u + v = 0 \)
(V5) For all \( \alpha \in \mathbb{R} \), and all \( u, v \in V \), \( \alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \)
(V6) For all \( \alpha, \beta \in \mathbb{R} \), and all \( u \in V \), \( (\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot v \)
(V7) For all \( \alpha, \beta \in \mathbb{R} \), and all \( u \in V \), \( (\alpha \beta) \cdot u = \alpha \cdot (\beta \cdot u) \)
(V8) For all \( u \in V \), \( 1 \cdot u = u \)
(V9) For all \( u, v \in V \), \( \langle u, v \rangle = \langle v, u \rangle \)
(V10) For all \( u, v, w \in V \), \( \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \)
(V11) For all \( \alpha \in \mathbb{R} \) and all \( u, v \in V \), \( \langle u, \alpha \cdot v \rangle = \alpha \cdot \langle u, v \rangle \)
(V12) For all \( u \in V \) other than \( 0 \), \( \langle u, u \rangle > 0 \)
For convenience, we’ll often omit the dot when doing scalar multiplication: i.e., we’ll write $\alpha v$ to mean $\alpha \cdot v$.

A set of vectors $\{v_1, \ldots, v_n\}$ are said to be linearly independent if the only solution to

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Equivalently, a set of vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if one of them, say $v_i$, can be expressed as a sum of the others, i.e. in the form:

$$v_i = \sum_{j \neq i} \beta_j v_j$$

Any vector space comes with some dimension: the largest number $n$ such that one can find a set of $n$ linearly independent vectors. The inner product defines a norm: given any $v \in V$, its norm $||v||$ is $\sqrt{\langle v, v \rangle}$. Finally, the so-called automorphism group of an $n$-dimensional vector space $V$ (the collection of transformations that preserve the structure of $V$) is the group $O(n)$, of rotations and reflections.

$$\langle u, v \rangle = ||u|| ||v|| \cos \theta$$  \hspace{1cm} (1)

where $\theta$ is the angle between the two arrows.

**Problem 1.** Prove that for all vectors $u \in V$, $u + (-1)u = 0$. Prove that for any other vector $v \in V$, if $u + v = 0$ then $v = (-1)u$.

The vector $(-1)u$ is referred to as the inverse of $u$; it will be abbreviated as $-u$. Given two vectors $u$ and $v$, we will sometimes abbreviate $u + (-v)$ as $u - v$.

**Problem 2.** Prove that for all vectors $u \in V$, if $u + u = u$, then $u = 0$.

**Problem 3.** Prove that for all vectors $u \in V$, and all $\alpha \in \mathbb{R}$,

(i) $0 \cdot u = 0$

(ii) $a \cdot 0 = 0$

**Problem 4.** Show that for all $u \in V$, $\langle 0, u \rangle = 0$. 

Metric affine spaces

Formally, a metric affine space $A$ comprises the following data:

- a set $A$ (of points)
- a metric vector space $V = (V, 0, +, , , , )$ (of displacements)
- a binary operation $- : A \times A \to V$ (subtraction)

The structure $A = (A, V, -)$ must obey the following axioms:

(A1) For all $a \in A$, and all $u \in V$, there is a unique point $b \in A$ such that $b - a = u$

(A2) For all $a, b, c \in A$, $(c - b) + (b - a) = c - a$

Using the above, we can define a binary operation of addition, $+: A \times V \to A$: given any $a \in A$ and any $u \in V$, $a + u$ is the unique point $b \in A$ such that $b - a = u$. The notation isn’t great, since we’ve used the same symbol (“+”) to denote both the map $V \times V \to V$ above, and the map $A \times V \to A$ introduced here. But there isn’t an ambiguity in practice, for the following reason: if you want to know which map a given occurrence of “+” refers to, just look at the symbols on either side of it. If they’re both vectors (e.g. $u + v$), then it’s referring to the map $V \times V \to V$. If one’s a point of an affine space and the other’s a vector (e.g. $a + v$), then it’s referring to the map $A \times V \to V$. Other treatments of metric affine spaces (such as David Malament’s notes) take the addition operation as primitive, and define subtraction in terms of it.
Space, time and motion

Newton, *De Gravitatione*:

[...] it is first of all to be shown that when a certain motion is finished it is impossible, according to Descartes, to assign a place in which the body was at the beginning of the motion; it cannot be said whence the body moved. And the reason is that according to Descartes the place cannot be defined or assigned except by the position of the surrounding bodies, and after the completion of a certain motion the position of the surrounding bodies no longer stays the same as it was before. For example, if the place of the planet Jupiter a year ago be sought, by what reason, I ask, can the Cartesian philosopher define it? Not by the positions of the particles of the fluid matter, for the positions of these particles have greatly changed since a year ago. [...] Truly there are no bodies in the world whose relative positions remain unchanged with the passage of time, and certainly none which do not move in the Cartesian sense: that is, which are neither transported from the vicinity of contiguous bodies nor are parts of other bodies so transferred. And thus there is no basis from which we can at the present pick out a place which was in the past, or say that such a place is any longer discoverable in nature. [...] And so, reasoning as in the question of Jupiter’s position a year ago, not even God himself could define the past position of any moving body accurately and geometrically now that a fresh state of things prevails, since in fact, due to the changed positions of the bodies, the place does not exist in nature any longer.

Newton, *Principia Mathematica*:

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.
Subspaces and product spaces

For any vector space $V$, a subset $W \subseteq V$ is closed under addition and scalar multiplication if both the following conditions are met:

(S1) For all $u, v \in V$, if $u \in W$ and $v \in W$ then $u + v \in W$

(S2) For all $u \in V$ and $\alpha \in \mathbb{R}$, if $u \in W$ then $\alpha u \in W$

A subspace $W$ of $V$ is a vector space whose vectors are a subset of $V$ closed under addition and scalar multiplication.

Problem 5. Show that the intersection of any non-empty set of subspaces of $V$ is a subspace of $V$.

For any affine space $A$ with associated vector space $V$, any subspace $W$ of $V$, and any $a \in A$, we define the affine subspace of $A$ through $a$ determined by $W$ as the affine space whose set of points is

$$\{a + u | u \in W\}$$

i.e., as all those points that can be “reached” from $a$ by application of members of $W$.

Given two metric vector spaces $V_1$ and $V_2$, we define their product $V_1 \times V_2$ as the vector space specified as follows:

- The set of vectors is $V_1 \times V_2$, i.e. the set of pairs $(u_1, u_2)$ where $u_1 \in V_1$ and $u_2 \in V_2$
- The zero vector is the pair $(0_1, 0_2)$, where $0_1$ and $0_2$ are the zero vectors from $V_1$ and $V_2$ respectively
- The addition of two vectors is defined by $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$
- The scalar multiplication of a vector is defined by $\alpha(u_1, u_2) = (\alpha u_1, \alpha u_2)$
- The inner product is defined by $\langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle$

Problem 6. Show that this does indeed define a metric vector space: i.e., that the structure $V_1 \times V_2$ so defined satisfies the axioms (V1)–(V12) above.

Given two metric affine spaces $A_1$ and $A_2$, we define their product $A_1 \times A_2$ as the metric affine space specified as follows:

- The set of points is $A_1 \times A_2$, i.e. the set of pairs $(a_1, a_2)$ where $a_1 \in A_1$ and $a_2 \in A_2$
- The vector space is $V_1 \times V_2$
- Subtraction is defined by $(a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$

Problem 7. Show that this does indeed define a metric affine space: i.e., that that the structure $A_1 \times A_2$ so defined satisfies the axioms (A1)–(A2) above.